The Failure of County Representation in the Western States

This study analyzes the impossibility of local jurisdictionally-based apportionment and the use of local jurisdictions in district planning. The analysis reveals the failure of county representation is generated by the distribution of population for purposes of legislative apportionment in the Western States. The basic result demonstrates the conditions under which county districts satisfy apportionment criteria from those where the choice of a district plan violates conditions using local jurisdictional units. These results describe the existence of political equilibrium in the form of integer solutions in both district magnitude *and* size of the legislature by allocation of varying delegation sizes for any number of remainder districts. The findings indicate the number and location of the proportion of counties either less than .4 or between a .4 and .5-population ratio for zero representation of counties where district allocation by local jurisdiction may be considered broken in terms of legislative apportionment. Integer findings are used to explain apportionment formulas, district plans, and weighted voting solutions in county units, population classification, district mapping, and either modified population ratios or population weights.

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Keywords
apportionment (elections) law, redistricting politics, district planning, local jurisdiction, linear programming model

The failure of county representation may best be summarized as the result of population concentration and growth in the largest districts, combined with an increasing number of small counties attaining less than a full population ratio. In The States, successful county representation exists where all counties attain a full population ratio, with at least one district allocated per-county or local jurisdiction. In the absence of exact population ratios, this implies a decision rule equating county population shares and apportionment or district allocation.

Based on previous findings, House and Senate apportionment was generally limited to low ratios for an initial or single district allocation, with larger ratios for any apportionments of additional districts to a local jurisdiction. As an example of this decision, consider a House apportionment based on small county shares, ranging from >0 to 0.333% of a State's population. Then compare these to county populations ranging from 0.333 to 1.75% population shares of the State's population. In this setting, the smallest counties would generally have been consolidated into multi-county districts, with two or more other counties (MC $\geq$ 2). At least one district would then have been allocated to the rest of the counties with a .333 to 1.75 population share.

This finding indicates both a single and multi-county solution in county representation for apportionment to local jurisdiction. More generally, county representation fails or begins to indicate failure when the population ratios attain between 0.4% and 0.5% population shares. Expanding the previous example, consider the following population shares: 1) 0 to .400, 2) .400 to .500, 3) .500 to 1.50, 4) 1.50 to 2.00, 5) 2.00 to 2.50, 6) 2.50 to 3.50, and 7) 3.50 and above. Given these population shares, apportionments of zero and one are granted to categories one and three, with the decision to apportion a full district allocation to category two indeterminate.

As shown by this example, the decision to apportion a second or third districts are also somewhat indeterminate, because these districts could be granted for population shares greater than 1.5% and 2.5% or for those greater than 2.5% and 3.5%. If the goal is to provide an apportionment solution, with the set of apportionment equal to {1, 2, 3}, then the apportionment from 1 to 3 districts allocated provides an integer solution for county representation. However, if the goal is to guarantee at least one district allocation, and then provide for more stringent requirements for additional representation then this Georgia solution consists of 1) 0 to .400 , 2) .400 to 1.75, 3) 1.75 to 2.50, 4) 2.50 and above with apportionments of {0, 1, 2, 3}. This county district plan allocates both single and multi-county districts, with single and multi-member districts. Extending the apportionment range to eight districts or more, describes House apportionment and most Senate district allocations, including any House or Senate county division districts.

Inasmuch county representation fails when there are a large proportion of small counties and/or indeterminate apportionment and district allocations. County representation succeeds when all of the counties exceed a full population ratio. The success of county representation involves a statewide commitment to uniform development, and more even development of metropolitan areas or less concentration of population in one or only a few urban areas. Either the organization of a small number of counties, or a large size of the legislature relative to the number of counties is also conducive for the success of counties attaining a full population ratio. In States where all or almost all of the counties attain a full population ratio, the apportionment of at least one district may be allocated per-county. In some situations this may require the formation of one multi-county district or a very small number of multi-county districts.

In comparison to House Apportionment, the evolution of County Senate District Plans implies some combinations of counties produce a County Senate District allocation that attains full population ratios for all of the apportioned districts. Senate Apportionment is therefore differentiated from House District Plans that attain full population ratios with (single) county representation. In this setting, County Senate Districts involve combinations of single and multi-county districts, and may also vary by single and multi-member districts. This combination of counties generates a district plan with greater numbers of single member multi-county districts and single county multi-member districts. The evolutionary stable strategy in these County Senate District Plans is to provide Senate apportionment and district allocation in varying numbers of (SC, SMD), (SC, MMD), and (MC, SMD) districts.

In the Western States, the failure of county representation varies from large numbers of small counties to only a few or no counties under a full population ratio. County Senate Districts evolved from generally one of two apportionments and district allocations. First, some Senate Apportionment allocated one district to each county. These district allocations only included organized counties, and city-county consolidated districts, with additional counties formed during the 20th century. Second, many States, including those in the Western State Legislatures, apportioned a size of the Senate equal to ½ the number of counties. This apportionment solution provides for a multi-county (MC = 2) district allocation of one district for combinations of two counties. In most States, this County Senate District Plan evolved into a combination of single and multi-county district allocations varying in the number of counties combined into Senate districts. As a consequence, this produced a General Apportionment equal to House (SC, MMD = 2) and Senate [(SC, MMD), (SMD, MC = 2)] district allocations.

Analysis of County Representation and Senate District Plans

Definition 1.0 number of districts ≡ D = {1, ..., m}, a finite integer set.

Definition 2.0 size of the legislative chamber ≡ N = {1,..., n}, a finite integer set.

Definition 3.0 number of local jurisdictions ≡ J = {1,..., j}, a finite integer set.

Theorem 1.0 Choice of the number of districts and a district plan = 24 solutions.

SMD	MMD	SC	MC	Equilibrium
0	0	0	0	Statewide
0	1	0	0	MMD
0	0	1	0	SC
0	0	0	1	MC
0	1	1	0	MMD + SC
0	0	1	1	SC + MC
0	1	0	1	MMD + MC
0	1	1	1	MMD + SC + MC
1	0	0	0	SMD
1	1	0	0	SMD + MMD
1	0	1	0	SMD + SC
1	0	0	1	SMD + MC
1	1	1	0	SMD + MMD + SC
1	0	1	1	SMD + SC + MC
1	1	0	1	SMD + MMD + MC
1	1	1	1	SMD + MMD + SC + MC

Proof.
permute([SMD, MMD, SC, MC]) ; solutions = 24;

[[SMD, MMD, SC, MC], [SMD, MMD, MC, SC], [SMD, SC, MMD, MC], [SMD, SC, MC, MMD], [SMD, MC, MMD, SC], [SMD, MC, SC, MMD], [MMD, SMD, SC, MC], [MMD, SMD, MC, SC], [MMD, SC, SMD, MC], [MMD, SC, MC, SMD], [MMD, MC, SMD, SC], [MMD, MC, SC, SMD], [SC, SMD, MMD, MC], [SC, SMD, MC, MMD], [SC, MMD, SMD, MC], [SC, MMD, MC, SMD], [SC, MC, SMD, MMD], [SC, MC, MMD, SMD], [MC, SMD, MMD, SC], [MC, SMD, SC, MMD], [MC, MMD, SMD, SC], [MC, MMD, SC, SMD], [MC, SC, SMD, MMD], [MC, SC, MMD, SMD]].

permute([SMD, MMD, SC, MC], 2) ; solutions = 12;

[[SMD, MMD], [SMD, SC], [SMD, MC], [MMD, SMD], [MMD, SC], [MMD, MC], [SC, SMD], [SC, MMD], [SC, MC], [MC, SMD], [MC, MMD], [MC, SC]].

permute([SMD, MMD, SC, MC], 3); solutions = 24;

[[SMD, MMD, SC], [SMD, MMD, MC], [SMD, SC, MMD], [SMD, SC, MC], [SMD, MC, MMD], [SMD, MC, SC], [MMD, SMD, SC], [MMD, SMD, MC], [MMD, SC, SMD], [MMD, SC, MC], [MMD, MC, SMD], [MMD, MC, SC], [SC, SMD, MMD], [SC, SMD, MC], [SC, MMD, SMD], [SC, MMD, MC], [SC, MC, SMD], [SC, MC, MMD], [MC, SMD, MMD], [MC, SMD, SC], [MC, MMD, SMD], [MC, MMD, SC], [MC, SC, SMD], [MC, SC, MMD]].

Definition 4.0 $\mathcal{P} \equiv$ polynomial.

Definition 5.0 Q ≡ quotient.

Definition 6.0 R ≡ remainder.

Theorem 2.1 N = Q + R.
Proof. $\mathcal{P}$ = Q + R. N = $\mathcal{P}$. N = Q - R. Assume fixed Q. R ≡ f(D) ≡ remainder district allocation.

Theorem 2.2 (Modulo solution) $\mathcal{P}$ = Q modulo R.

Theorem 2.3 Remainder district solution, N = Q + RMD.
Proof. N = Q modulo R. RMD = f(D) = flotorial district allocation. If a polynomial P(N) is divided by N - R, the remainder is R(D) = F[r].

Theorem 3.1 N = Q + R = J.

Proof. P(N) = Q(J). Assume P(N) > Q(J). P(N) + R(N) > Q(J) + R(N). P(N) + R(N) > Q(J) • R(N), for R(N) > 0. P(N) • R(N) < Q(J) • R(N), for R(N) < 0.

Verification. From simulation of House Apportionment in the Western States, with population ratios from 1990 Census. **Lemma 1.0**.

Idaho
86/84
-2 district remainder
Arizona
61/60
-1 district remainder
Hawaii,
51/51
0 district remainder
New Mexico
70/70
0 district remainder
Wyoming
64/64
0 district remainder
Nevada
41/42
1 district remainder
Utah
74/75
1 district remainder
Washington
97/98
1 district remainder
Oregon
56/60
4 district remainder
Montana
96/100
4 district remainder
California
75/80
5 district remainder
Colorado
58/65
7 district remainder

Theorem 3.2 N = Q = J.

Proof. A polynomial P(N) has a factor N - R if and only if P(R) = 0.

Verification. Size of the Legislatures divided by the number of Counties, 1900-1990, N = J, Senate Apportionment in The States.

Arizona = 1.746
Connecticut = 4.300
Delaware = 6.000
Idaho = .950
Hawaii = 4.861
Maine = 2.019
Maryland = 1.388
Massachusetts = 2.857
Montana = .973
Nevada = 1.169
New Hampshire = 2.400
New Jersey = 1.271
New Mexico = 1.003
New York = .883
North Dakota = .923
Oregon = .846
Rhode Island = 8.760
South Carolina = 1.000
Utah = .806
Vermont = 2.143
Washington = 1.159
Wyoming = 1.319.

Definition 7.0 D ≡ divisor.

Theorem 3.3 N = Q•D + R.

Proof. P = Q + R. P = (D•Q) + R. P/D = Q + R/D, D ≠ 0. 0 ≤ R < D. P = (Q•D) + R, 0 ≤ R < D.

Verification. Size of the Legislatures divided by the number of Counties., 1900-1990, N = ½•J, Senate Apportionment in The States.

Alabama = .5233
Arkansas = .4627
Illinois = .5243
Indiana = .5435
Iowa = .5051
North Carolina = .5026
Wisconsin = .4635.
California = .6909
Colorado = .5642

Theorem 3.4 N = Q + R = J + AR.

Proof. P(N) / D(N) = Q(N) + R(N) / D(N), where R(N) = 0 or where degree of R(N) < degree of D(N).

Lemma 1.0 Range solution ≡ (N, σ).

Proof. Range ≡ σ = λ•d = [0, $\underline{\sigma}$].

Lemma 2.0 Density solution ≡ (N, δ).

Proof. Density ≡ δ = λ•j = [0, $\underline{J}$] = $\mathscr{T}^*$.

Theorem 4.0 Range and density solution ≡ σ•δ = Å.

Proof. **Lemmas 1.0 & 2.0**. Range and density = (N, σ•δ).

Lemma 3.0 (Group decision function) Γ = G(N, σ•δ).

Lemma 4.0 Full Quota Districts = Greatest Integer[population ratio].

Proof. Delegation size ≡ d. The largest delegation size ≡ maximum $\underline{d}$. I = {d}, a set of district sizes, equal to Int[s•N] = Å.

Lemma 5.0 Distribution of population ratios ≡ population shares • sizes of The Legislatures.

Proof. s•N = d_m ≡ district magnitude. d_m = P[r].

Theorem 5.0 N = Σ(σ•δ) = λ•B^2 = D ≡ District allocation plan.

Verification. House Apportionment Simulation using 1990 Census data.

Arizona
2
9
2
2
15
15 = **4**(0) + 3(1) + 6(2) + 1(11) + 1(35) = 60

California
3
21
3
31
58
58 = **34**(0) + 11(1) + 5(2) + 3(3) + 2(4) + 1(6) + 1(7) + 1(24) = 80

Colorado
5
9
5
44
63
63 = **49**(0) + 4(1) + 2(2) + 1(3) + 2(4) + 1(5) + 2(8) + 2(9) = 65

Hawaii
4
4
4 = **1**(2) + 1(5) + 1(6) + 1(38) = 51

Idaho
4
30
0
10
44
44 = **10**(0) + 20(1) + 5(2) + 3(3) + 1(4) + 3(6) + 1(8) + 1(17) = 84

Montana
6
31
2
17
56
56 = **19**(0) + 23(1) + 5(2) + 2(3) + 1(4) + 2(6) + 1(7) + 2(10) + 1(14) = 100

Nevada
2
6
1
8
17
17 = **9**(0) + 6(1) + 1(9) + 1(26) = 42

New Mexico
2
22
2
7
33
33 = **9**(0) + 11(1) + 5(2) + 4(3) + 1(4) + 1(5) + 1(6) + 1(22) = 70

Oregon
4
16
4
12
36
36 = **16**(0) + 11(1) + 3(2) + 1(3) + 1(5) + 2(6) + 1(7) + 1(12) = 60

Utah
4
12
3
10
29
29 = **13**(0) + 9(1) + 2(2) + 1(3) + 1(7) + 1(8) + 1(11) + 1(32) = 75

Washington
5
21
1
12
39
39 = **13**(0) + 14(1) + 3(2) + 2(4) + 1(5) + 1(7) + 1(9) + 1(12) + 1(30) = 98

Wyoming
4
18
0
1
23
23 = **1**(0) + 8(1) + 5(2) + 3(3) + 2(4) + 2(5) + 1(9) + 1(10) = 64

Analysis of the Failure of County Representation in the Western States

Using the results from **Theorems 3.1 & 5.0**, House apportionment maps are constructed from the simulation data and reported in **Appendix I**. First, population shares are calculated from the 1990 Census with county population shares equal to county population divided by the reported total State population. Second, applying **Lemma 5.0** generates a distribution of county population ratios equal to the population share multiplied times the size of the legislative chamber. Third, as derived from **Lemma 4.0**, the greatest integer function is used to simulate an apportionment to counties in finite delegation sizes. Forth, the local jurisdictions are classified a county representation failure, in the sense that these counties fail to attain a full population ratio; the numbers of zero district allocations are counted and reported as verification of **Theorem 5.0**. As a consequence, these counties would *not* be allocated at least one district by single county population ratios, using this House apportionment simulation, without apportionment and district allocation modifications.

The apportionment maps indicate four categories of district allocation:

- blue = delegation size > 5, largest county district allocations
- white = 1- 4 delegation size range of district allocation
- yellow = $.4 < \lambda < .5$ range of population ratios
- red = $\lambda < .4$ county representation failures.

Because population ratios are used to determine delegation sizes, the largest delegation sizes are reported as an integer, for House apportionments consisting of greater than five districts. The largest districts generate variation in delegation sizes, and vary by State, in the proportion of the seats or positions by size of the legislative chamber. Amongst the States, the largest allocations consist of the fewest number of counties, with delegation sizes greater than five districts.

Once the largest districts are allocated, all districts within a one to four range in population ratios are assigned an integer delegation size. These districts may be considered the medium sized delegations, given the distribution of district magnitudes constructed from population ratios. As a result, these district allocations indicate successful county representation for the largest and medium delegation sizes. For these district allocations, single county apportionment guarantees at least one district per-county.

The counties below a full population ratio indicate failure in county representation, and the apportionment to these counties involves modification of single county representation. The mapping provides a description of county representation failure into two categories. The first is generally the whole group of counties below a full ratio by location in each Western State. The second category involves those counties that were marginally below integer classification for single county district allocation. Before the 1950 or 1960 Censuses, these counties may have qualified for a single county district by House apportionment and district location plans.

In summary, the counties between .4 and .5 of a population ratio describe marginal apportionment and district allocation planning decisions. These counties fail to qualify for a full single county district, even though some counties may have attained a single county district at some point during apportionment and redistricting. Additionally, these counties were frequently combined with other counties in County Senate District Plans, so there are many examples of multi-county districts. The marginal counties are therefore more likely to be combined with other counties, and given the population ratios of the other counties, this becomes a spatial model of apportionment and district allocations. The placement effect of these counties describes the potential combination of these counties with the other three categories by district magnitude.

For the purposes of House Apportionment, the design of county representation implies at least one seat or position and single county district allocation. Among the Western States in the 1990 Census, only Hawaii's four consolidated, counties had at least one district apportionment with single county district allocation. The mapping results indicate Wyoming had only one county (Niobrara) at less than a full population ratio, and no marginal counties with population ratios between .4 and .5. The Arizona mapping reveals four counties below a full ratio, with two marginal counties (Graham, Santa Cruz) and two small counties (La Paz, and Greenlee). This result indicates a 26.67% failure in single county representation.

Even so, the Arizona map suggests the potential for single county combinations into multi-county districts. For the smallest two counties, La Paz has been combined with the largest district (Maricopa County), and could potentially be combined with either a large or medium sized district. Greenlee is adjacent to medium sized districts and a single marginal county (Graham County) with which it has been combined with before to form a district. Graham County is adjacent to counties in all three of the other categories of population ratios, indicating potential combinations with large, medium, or smaller districts. Santa Cruz is located next to a large district (Pima County) and a medium district (Cochise). In 1990, the simulation data indicates county representation success with three multi-county (MC = 2) apportionments consisting of La Paz - Maricopa, Santa Cruz - Pima, and Graham - Greenlee.

In Nevada, the findings indicate two largest district counties and six counties with medium delegation sizes. As the only marginal population ratio, Humboldt County is adjacent to a large, medium, and smaller counties. The smaller counties have been combined into regional districts equal to the combination of a large number of counties ($MC \geq 4$). As the delegation sizes in Clark and Washoe counties increased (from 1900 to 1990), the remaining medium sized counties each consist of single county district allocations. As the mapping reveals, the other eight smaller counties and the single marginal county could be combined into a single regional district. Other combinations are also possible with the largest districts and the medium districts, suggesting varying multi-county (MC $\geq$ 2), single district allocations.

The New Mexico map reveals seven smaller counties and two marginal counties: Torrance and Sierra. The location of these smaller counties suggests pairings and combinations with other smaller counties for pursuit of district allocation. Additionally, both marginal counties are located adjacent to large, medium, and smaller counties. In previous House Apportionment and District Plans, these combinations generated flotorial, multi-county districts. Given a population ratio equal to 4.57, Santa Fe is also included among the medium sized counties so that the mapping results indicate multi-county medium sized solutions, varying in the number of small counties (MC $\geq$ 2).

The county population ratios in Montana indicate two marginal counties (Broadwater and Mineral) and seventeen smaller counties. For the two marginal counties, the mapping of the delegation sizes suggests possible two and three category combinations with the largest counties, and medium sized or smaller counties. This mapping also reveals medium sized counties distributed throughout Montana, with the larger counties in western Montana and smaller counties located in eastern Montana. These results imply multi-county solutions among the smaller counties, with some potential for combinations with either the largest or medium sized counties. The mapping also indicates six larger districts, with delegation sizes estimated to range from six to fourteen seats or positions in House Apportionment.

In the 1990 Census, none of the Idaho counties are estimated to be in the marginal county category, with a population ratio between .4 to .5 for purposes of district allocation. There were ten smaller counties, with population ratios less than .4 of an apportionment. The location of these counties suggests either consolidation of smaller counties into individual districts, or some combinations with either the larger or medium sized counties. The Idaho map reveals five larger districts located in three distinct locations, with medium sized counties distributed throughout the State. The mapping indicates the smaller counties may be paired with either medium or larger counties, as additional possibilities to consolidation of a large number of counties into individual districts.

The results for Washington indicate only one marginal county (Jefferson County) and twelve smaller counties. By mapping population ratios, most of the smaller counties are located in eastern Washington, surrounded by either medium sized counties or one of the five largest districts (Spokane County). The three largest districts are located in western Washington, with medium sized counties distributed throughout the State. These findings suggest multiple multi-county combinations of the smaller counties for district allocation, and additional pairings of the smaller counties with a larger district or medium sized counties. This map also reveals the three largest counties are contiguous to each other and surrounded by medium sized counties.

The mapping of Utah population ratios indicates three marginal counties and ten smaller counties. As shown in the map, medium sized counties are distributed throughout the State. Additional results reveal seven of these smaller counties and two of the marginal counties are located in southern Utah, so that two of the three marginal counties have possible combinations with smaller or medium sized counties. Among the marginal counties, only Wasatch County is adjacent to the largest and medium sized counties. As indicated by the mapping, all four of the largest counties are contiguous and located in central to northern Utah.

The Oregon mapping indicates a somewhat greater regional effect, with most of the twelve smaller counties and two of the four marginal counties located in eastern Oregon. The three largest districts are contiguously located in western (or northwestern) Oregon. Among the four marginal counties, Wasco County is adjacent to one of the largest counties, six smaller counties, and one of the medium sized counties. Union County is adjacent to three smaller counties and one medium sized county. Tillamook County is adjacent to four medium sized counties and one of the largest counties. Lastly among the marginal counties, Curry County is adjacent to three medium sized counties.

The California map describes a regional effect with all three of the largest counties in Southern California, with thirty of the thirty-one smaller counties, and the three marginal counties in Northern and Central California. Among the three marginal counties, Butte County is surrounded by smaller counties, with Merced and Placer counties adjacent to both medium sized and smaller counties. The spatial distribution of medium sized counties ranges from those in the Bay Area to Southern California, with pairings of medium, marginal, and smaller counties in Northern California. With the exception of Imperial County, the pairings in Southern California involve the largest districts and medium sized counties. In California (and Nevada), smaller counties constitute more than a majority of the counties, at 58.6% and 52.94%.

The Colorado mapping reveals forty-four smaller counties and five marginal counties. The five marginal counties and the smaller counties are spatially distributed throughout the State. Even though there are fewer medium sized counties, these are also distributed throughout the

State. Among the largest five districts, four of the five counties are contiguous and located in eastern Colorado. As the mapping reveals, more than 77.78% of the counties attained less than a full population ratio in the 1990 Census. The four of five marginal counties, consisting of Eagle, Otero, Delta, and Montrose counties, had potential pairings with either medium or smaller counties. As shown in the mapping of population ratios, Morgan County had possible combinations with a large district, a medium sized county, or from one to two smaller counties.

These State findings indicate flexibility in the possible combinations or pairings of marginal and smaller counties with the largest districts and medium sized counties. As these results suggest, there are some complications in the number of counties required to attain population ratios. These results imply the extent of county representation failure varies by State, with 1) some states closer to single county representation with at least one House apportionment seat or position, 2) a plurality or majority of counties attaining full population ratios, and 3) some states with only multi-county solutions possible varying in the number of counties per-district. As demonstrated by these results, the Western States vary by the location and number of the largest district allocations. These States also vary by the location, number and proportion of smaller counties indicating greater potential for regional effects and single district allocations to a large number of counties ($MC \geq 4$ counties). By choice of district plans, these results imply House Apportionment and district allocations' equal to choice among SC, SC + MC, SC + MC(2) + MC(3) + MC(4) + MC(>5) plans. The findings derived from **Theorem 1.0**, **Theorem 3.1**, **Theorem 5.0**, and the mapping of smaller and marginal counties provide support for maintaining county boundaries intact and therefore minimizing the number of county division districts.

Examples of House and Senate Apportionment and District Allocation Plans

The previous results demonstrate the potential for SC, and SC + MC apportionment and district allocation. By 1990, the courts declared the era of county representation over with the demise of each county in Wyoming attaining at least one district. The findings indicate an ordering of Western States where the location and existence of smaller counties generate failure in single county representation with SC district allocation. The extent of this failure varies by location of the smaller counties, as shown in the mapping of population ratios. The proportion below a full quota provide some indication of the extent of failure, with State ease at guaranteeing at least one district allocation and county representation varying from Hawaii (0.0%), Wyoming (4.35%), Arizona (26.67%), Nevada (52.94%), New Mexico (27.27%), Montana (33.93%), Idaho (22.73%), Washington (33.33%), Utah (44.83%), Oregon (58.62%), California (58.62%), to Colorado (77.78%). As a consequence, these results describe a transition from SC apportionment and district plans to SC + MC apportionment and district allocations. Inasmuch the 1990 estimate a varying attainment of single county (SC) House apportionment and district plans, guaranteeing at least one district allocation by population ratio.

More generally, the group decision function summarizes range and density solutions for apportionment and district allocation plans. These findings are measurable in integer classification of population ratios used to derive finite delegation sizes. In the case of Wyoming the failure of single county representation involves only one county to attain a full population ratio. By single county consolidation, the largest district solution equals 23 = **9(1)** + 5(2) + 3(3) + 2(4) + 2(5) + **2(9)** = 64. By substituting district magnitude for delegation size, a partial representation solution equals 23 = **1(.5) + 7.5(1)** + 5(2) + 3(3) + 2(4) + 2(5) + 1(9) + 1(10) = 64.

Theorem 6.0 (District Plan by number of Districts) $\mathcal{P} = \lambda \bullet B^2 = D$.

Theorem 7.0 $\mathcal{P} = (\mathcal{P}\,; Q) = N = N \bullet [Pr(SMD) + Pr(MMD)]$.

Theorem 8.0 Given a fixed size of the legislature N, $I = \mathcal{P} = D$.

Theorem 9.0 $N = \Sigma(\sigma \bullet \delta) \equiv$ Apportionment formula.

Theorem 10.0 $N = \Sigma(\sigma \bullet \delta) = f(J) \equiv$ Fragmentation solution.

Lemma 6.0 (Proportional district magnitude) d_m = population ratio.
Proof. $d_m = s \bullet N$. $s \bullet N = P[r]$.

Lemma 7.0 Weighted voting scheme = population ratios.
Proof. $s \bullet N = P[r]$. $\text{Å} = P[r]$, population weighting scheme = distribution of population ratios. $P[r] = d_m = \text{Å}$. $P[r]$ = population weighted voting.

Proposition 1.0 Apportionment formula in fixed size of the legislature and a finite integer delegation size.
Proof. $\text{Å} = [N, \sigma]$.

Proposition 2.0 Apportionment mapping.
Proof. $\text{Å} = \phi(J)$.

Proposition 3.0 Delegation size or district magnitude.
Proof. $s \bullet N = d$. $d = \phi(D)$.

Proposition 4.0 District allocation.
Proof. $D = \phi(N)$.

Proposition 5.0 Apportionment to local jurisdiction.
Proof. $\text{Å} = d = \phi(J)$

Proposition 6.0 Reapportionment = change in the district size.
Proof. Δ = finite change or difference. Change in delegation size $\equiv \Delta d$.
$\Delta d = \phi(J)$.

Proposition 7.0 District Plan.
Proof. $d = \lambda \bullet B^2 = \phi(D)$, in delegation size, district boundaries, and number of districts.

- ☐ number of districts (D)
- ☐ range of delegation sizes $\sigma(d)$
- ☐ district boundaries, area ($\lambda \bullet B^2 = \mathcal{L}(B)$).

Proposition 8.0 Redistricting Plan.

Proof. $\Delta d = \lambda \bullet B^2 = \phi(D)$, change in delegation size, numbers of district boundary changes, and number of districts or district allocation.

- changes in a district plan
  - changes in the number of districts
  - changes in the district numbers, letters, names
- change in the combination of local jurisdictions
  - status quo description of district boundaries
  - description of changes to the status quo (100% of the districts changed)
  - number of reform alternatives
  - number of boundary decisions
- description of district boundaries, number of boundary changes

Proposition 9.0 Local division.

- ■ number of local jurisdictions
- ■ local jurisdictional boundaries
- ■ fragmentation solution

Proof. $J = \mathfrak{L}(B) = \mathscr{F}^*$.

Proposition 10.0 District Planning Congruence with Local Division.

Proof. $\lambda \bullet B^2 = J = \mathfrak{L}(B) = \mathscr{F}^*$.

- combinations of counties
- combinations of towns
- combinations of cities and counties
- combinations of towns, boroughs, parishes, counties, cities and city wards.

Proposition 11.0 District Plan Allocation with Local Division.

Proof. Senate ⇒ district # allocation. County ⇒ District. $N \rightarrow D$ & $J \rightarrow D$.

- district number
- lettered district (MMD)
- local jurisdictionally named district.

Proposition 12.0 Apportionment Plan.

Proof. $\{N, \sigma, \underline{d}\}$ = apportionment plan.

- fixed N, in size of the legislature
- $\sigma \equiv$ range in integer delegation size
- range solution $\equiv (N, \sigma)$
- $\underline{d} \equiv$ largest district size allocation.

Outside of the Western States, there are three pertinent examples of failures in county representation. Each example is derived from basic **definitions 1.0, 2.0, & 3.0** in numbers of districts, size of the legislature, and number of local jurisdictions. These State examples describe failures in Senate and House apportionment and district allocation plans in three Southern States: Tennessee, Georgia, and Alabama.

In the first example, the Tennessee Legislature attained a 33-member Senate and 99-member House in 1882. This size of the Legislature has remained fixed since this time with major reapportionments of county representation in 1891 and 1901. From statehood to 1882, 96 counties were formed in Tennessee, with one county, James County, consolidated with another small county prior to the 1901 apportionment and district allocation plan. For this plan, 99 House seats or positions were allocated to 95 counties providing for very few additional representation districts. Because some of the counties failed to attain full population ratios, complications arose in preventing any guarantee for each county to be apportioned at least one member in the House. As a result, the N = J + AR plans failed on a single county representation basis, with 99 = 95 single county districts + 4 additional representation districts.

Given the failure to attain full population ratios, multi-county (MC) solutions were required for the House Apportionment Plan with SC + MC districts. The solution was to design numbered districts for the SC + MC district plan. The district allocation plan involved both single and multi-county districts with varying numbers of counties per-district. This House Apportionment Plan also included both single and multi-member districts with varying MMD delegation sizes. At the time, the use of multi-county House districts was relatively infrequent among The States, with most States adopting single county representation plans.

From 1901 to 1961, changes occurred to many of the House Districts so that the district numbers ceased to exist. In some Districts, this involved changes in the number of counties included by either separating a House District into a single county district, or forming a multi-county House District from an increase in the number of counties included in the district allocation. In other Districts, the combinations of counties were changed generating a different pairing of counties or combination for the same number of counties. In each instance, the House District number ceased to exist providing for a new House District by location that was inconsistent with the 1901 House Apportionment and District Allocation Plan. By 1951 through 1961, numerous reallocation bills had been passed by the Tennessee Legislature, but these did not provide for a comprehensive renumbering and therefore new district allocation plans.

The failure to redistrict was described as caused by legislative inertia. The Legislature did enact new apportionment bills that provided for district reallocation of counties. Even so, the apportionment and district allocation was not for the purposes of single county representation. Instead the usage of a mixed representation plan created a range of district sizes for varying numbers of districts per-county. In 1900, the uses of low population ratios (e.g., a .400 population ratio) and a high population ratio for the second district (e.g., the use of a double ratio to determine the second apportionment, e.g., effectively a 1.75 or greater population ratios) guaranteed a House apportionment and district allocation plan with mostly single county, single member districts. The larger districts tended to be under-apportioned from the estimated population ratios providing for additional districts to be allocated on a single county basis. Lastly, the allocation of two and three county single member House districts also generated additional districts for single county representation and apportionment to the largest districts.

The 1900 Census data produced estimated population ratios in Tennessee closer to single county representation districts than succeeding later Censuses, inclusive of the 1990 data. From the 1901 to 1951 and 1961 House Apportionment and District Plans, the number of smaller counties not attaining a full population ratio increased from 20 (in 1900) to 45 counties by 1990. These results produced a decline from 79.8% to 54.6% single county, single member districts. By count, there were 65 single county, single member districts in 1900. This number reduces to 38 single counties attaining a full population ratio by 1990. The size of the largest district population ratios increased from 8 to 17 from 1900 to 1990, with fewer than 8 apportioned to the largest district in 1900.

On the Senate Apportionment, there were 33 Senate Districts allocated in 1901. Using the 1900 data for estimation, there were 88.5% multi-county districts and 84.8% single member districts. The numbers of multi-county Senate Districts remained approximately the same, at 87.4% in 1990, but the numbers of counties with more than one Senate District increased from 1900 to 1990 producing 39.4%, single member county subdivision districts in the largest counties. These results demonstrate the County Senate Plan of 1901 evolved differentially from multi-county districts, with two or three counties, to a County Senate Plan consisting of multi-county districts with varying numbers of counties and large single county districts, with multiple county subdivision districts. Like the House Apportionment, the changes in the combinations of counties implied the initial district numbers ceased to exist during the 1901 to 1951 and 1961 changes to the district allocation plan. As a consequence, the reapportionments of 2 districts each with 2 counties produced changes to 1 SC district for a larger county, and an MC = 3 district consolidating three smaller counties into a second single member district.

Any failure in county representation involved the increasing numbers of smaller counties, district elimination and a reduction in the number of single county districts. Reapportionment of county delegation sizes produced larger district allocations and larger numbers of counties in single member districts. These apportionment and district allocation plan changes were described as changes in Direct (county) representation. In Tennessee, indirect representation described multi-county, flotorial or partial districts and county subdivision district allocations to a few of the largest cities as city districts. Among the challenges of making Senate and House Apportionment, in Tennessee with N = {33, 99) and J = 95, the frequency of district elimination added complications by causing changes in combinations of counties and the number of districts.

The second example is generated by adopting a fixed range in delegation sizes, $\underline{\sigma} = \{1, 2, 3\}$ in a finite integer set of apportionments. The single county representation plans allocated from one to three delegates per-county, depending on their population classification as an urban, town, or rural county. Georgia had 161 counties until the 1930 Census, when three counties (Fulton, Campbell, & Milton) were consolidated for the purposes of local jurisdiction and legislative apportionment. Instead of single county, single member district allocation, the consolidation of these three counties produced a single county, multi-member district allocation and provided incentives for city-county consolidation. Any reductions in the number of counties also influences County Senate District Plans. As a result of the three county consolidation, Senate apportionment and district allocation increased the number of counties per-district. This metropolitan consolidation also produced changes in the combinations of counties, multi-member Senate Districts, and the range of SC + MC district allocations.

The third example, Alabama, had one additional county formed after the 1900 Census (Houston County). The formation of this county produced the 67 counties in Alabama. For the 1911 House Apportionment, the formation of a larger county resulted in a five and then sixth seat or positions increase, from (1900) N = 100 to (1902) N = 105 and then (1906) N = 106. Given the subsequent distribution of population ratios, House Apportionment was generally on a single county apportionment and district allocation plan until 1951 and 1961. Between the 1900 and 1950 Censuses, the largest district population ratio increased from 8 to 16 districts. As this expansion occurred, in Jefferson County, the House Apportionment remained at 4 districts and then expanded from 8 (1951) to 16 (1965). Generally speaking this example suggests that the requirements for additional representation were greater than for single county representation. This example also demonstrates a more limited range in the multi-member districts adopted than what is equated to population ratios.

These three examples describe failures in single county representation and the relationship between the number of counties and single county representation plans. By changing the number of counties, this produced changes in both Senate and House Apportionment. Any changes in the number of counties influenced the number of districts and district sizes, changed single county representation to multi-county district solutions, increased the range in numbers of counties' per-district, and produced changes in the relationship between the number of local jurisdictions and legislative apportionment.

Numerical examples describe the general relationship between the size of The State Legislatures and the number of (towns or) counties. For the purposes of apportionment and district allocation, assume there are 50 counties, 25 seats or positions in the Senate, and either 50 or 75 House Districts. Given a House Apportionment Plan allocates 50 seats or positions, this implies either a single county representation plan, with the guarantee of one seat or position per-county (N = J), or mixed representation plans with combination design of single and multi-county districts and single and multi-member districts. Assume the House Apportionment Plan allocates 75 seats or positions, then there is some potential to guarantee at least one district per-county, and additional representation for a single county representation plan that consists of a combination of single county, single and multi-member districts.

The N = 75 sized House may be sufficient to provide each of the J = 50 counties at least one district. Assume this is not feasible given the location and distribution of population ratios, so that there are too many smaller counties or the locations of the smaller and marginal counties are such that combinations of counties fail to produce contiguous and compact districts. Other possible combinations of sizes of the legislature may be considered, such as (50, 100), (40, 120), (75, 150), and (60, 180). These other combinations have been discussed as reform alternatives, and many of The State Legislatures have these combinations or approximately these combinations in sizes of the legislative chambers [(50, 100), {50, 120), (50, 150), (40, 100), (40, 120), (40, 150), (60, 150)]. As these numerical examples reveal, increasing the size of the legislature and sustaining a bicameral ratio is more complicated than changing the size of the legislature, number of districts, and even the number of counties to plan for House and Senate Apportionment and District allocation. Any effort to implement single county representation involves therefore the location and distribution of population ratios and may require some changes in the district allocation plans, number of districts, size of the legislature, and number of counties.

TABLE 1.1 State Reapportionment by changes in the Size of the Legislature, 1900-1990

State	Frequency*	Percent
AK	3	1.3
AL	2	.8
AR	1	.4
AZ	9	3.8
CT	10	4.2
DE	4	1.7
FL	7	2.9
GA	10	4.2
HI	2	.8
IA	2	.8
ID	9	3.8
IL	4	1.7
LA	6	2.5
MA	1	.4
MD	11	4.6
ME	4	1.7
MI	3	1.3
MN	6	2.5
MO	5	2.1
MS	4	1.7
MT	12	5.0
ND	8	3.4
NE	2	.8
NH	6	2.5
NJ	2	.8
NM	6	2.5
NV	11	4.6
NY	6	2.5
OH	14	5.9
OK	8	3.4
PA	4	1.7
RI	6	2.5
SC	2	.8
SD	5	2.1
TX	5	2.1
UT	11	4.6
VT	4	1.7
WA	6	2.5
WI	1	.4
WV	6	2.5
WY	10	4.2
Total	238	100.0

* Number of Senate or House Reapportionments

FIGURE 1.0

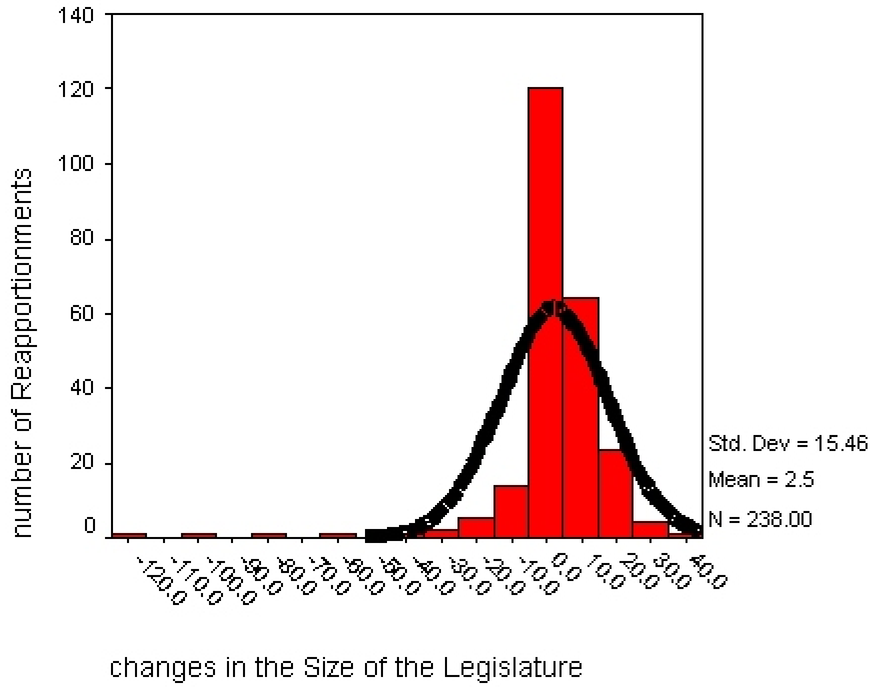


TABLE 1.2 Descriptive Statistics, State Reapportionment, 1900-1990

	N	Mean	Std. Error	Std. Deviation	Skewness	Kurtosis
REAPPORTIONMENT	238	2.47	1.00	15.46	-3.935	25.060

TABLE 1.3 Descriptive Statistics, State Reapportionment by Legislative Chamber

CHAMBER	Median	N	Mean	Std. Error	Std. Deviation	Skewness	Kurtosis
House	4.00	144	1.89	1.61	19.34	-3.288	15.849
Senate	2.00	94	3.36	.59	5.76	.983	4.453
Total	3.00	238	2.47	1.00	15.46	-3.935	25.060

TABLE 1.4 Test of Homogeneity of Variances, Legislative Chamber Variances in Reapportionment

Levene Statistic	df1	df2	Sig.
16.438	1	236	.000

From 1881 to 1901, legislative apportionment implied some construction of County Senate District Plans and House Apportionment equates single county representation plans. The County Senate Districts required design of a combination of single and multi-county districts, with more than 90% of the seats or positions elected by single member districts. District allocations of a second Senate District were infrequent and in some cases controversial decisions because this increased or changed the number of counties' per-Senate Districts. The largest Senate District allocations were seldom more than a multi-member district with two Senate Districts, and these comprise by far most of the multi-member Senate Districts allocated.

The large district exceptions are reported in **Appendix II** by 1900 population ratios. In 1898, for example, the Senate Apportionment = 10 and the House Apportionment = 25, by Ward Division = 17 in Orleans Parish, Louisiana. Other counties and city districts with Senate population ratios more than 1.5 are shown in **TABLE 1.0**, **Appendix II.** In many instances a second Senate District was not apportioned, producing a single county, single member district and a district available for use as a flotorial district or additional representation to reduce the number of counties' per-Senate districts.

The distribution of population ratios for House Apportionment was consistent with New England town representation. The House Apportionment and District allocation of at least one seat or position per-Town resulted in larger numbers of Districts. House Apportionment to Town Districts generated distributions of one and two seats or positions per-Town. As the Town populations increased, this produced greater numbers of multi-Town Districts and Town Districts with multiple allocations. In other States, single town representation produced apportionments and district allocations with the size of the legislature equal to the number of local jurisdictions.

In summary, the Alabama, J = 67 county's example generalizes from single county representation plans to House and Senate Apportionment and District allocation with varying numbers of counties per-district and mixed representation plans with single and multi-member districts varying in delegation size. For population ratios greater than .75, either House or Senate Apportionment produced a district allocation consistent with single county representation. As the distribution of population ratios increased in range, The States made single district apportionments using population ratios greater than .333 to at least 1.75. The existence of a greatest least upper bound suggests allocations from .400 to 1.75. By implication, any classification based on the greatest integer functions suggests a range in population ratios from .500 to 1.50. This analysis suggests population ratios less than .400 were not apportioned single county representation districts and likely required multi-county solutions for apportionment and district allocation. Instead these may have been assigned a zero representation value in an initial round of apportionment, and then consolidated into districts with other small and marginal counties. The examples in this section also strongly suggest that single county representation failed in the States because population ratios drifted outside of the $.75 < \lambda < 1.25$ range used for allocation districts on a single county basis. The choice to continue to guarantee at least one district-per county, by single county representation planning, reveals those population ratios less than .500, such as .400 and .333, exceeded the range-bound for the apportionment and district allocation of a single seat or position. As derived from these results, apportionment of a single district allocation ranged in population ratios from .5 to 1.75. These findings also indicate failure for single county representation plans in the .400 < population ratio < .500 range bound. This failure generalizes to County Senate Districts and House Apportionment and District allocation.

These state examples demonstrate instances of single county representation failure. The findings from the previous section describe spatial apportionment and provide a location analysis of district planning. The findings indicate a county to county pattern regional, in single county, largest districts, with uniform effects for explaining variation in population ratios. As shown by mapping, the relatively uniform distribution of medium sized and marginal counties have implications for the number of combinations of counties into contiguous and compact districts.

Some of state effects appear to have been

- redistribution from the largest districts
- adoption of large district solutions
- general apportionment and district allocation to medium sized counties
- district allocation plans for smaller and marginal counties
- use of flotorial districts or multi-county remainder districts
- extension of single-county solutions to contiguous counties
- frequent use of multi-county solutions with varying numbers of counties
- apportionment of new districts by changing the size of the legislature.

Using minimum and maximum conditions, the range-bounds for sizes of The Legislatures, number of districts, and number of jurisdictions explain the choice of Apportionment and District allocation plans and as a result implies changes derived from reapportionment and redistricting. From the Apportionment statements, the possibilities exist for attaining both constraints, a minimum constraint, a maximum constraint and greatest least upper bound—as demonstrated by the analysis of the range-bounds for smaller counties, marginal counties, medium sized and the largest districts, assuming no constraints on population ratios.

The Area Factor

The issue of area, in location and distance, involves a generalization of the size of the legislatures and the number of electoral districts. The area factor describes any relationship between legislative apportionment and local jurisdiction, including the compactness and congruence of district and local jurisdictional boundaries. The use of population and area factors are therefore explanations for what are variances, by location and distribution of population ratios, in contiguous apportionment *and* moiety district allocation plans.

In Michigan, a moiety clause was adopted that provided for a 4:1 ratio of population to area factors. This amendment was intended to provide each of $J = 83$ counties a weighting factor to modify the effects of apportionment and district allocation to a large number of small and marginal counties. In 1952, two Senate Districts were added, increasing the size of the State Senate from 32 to 34. The Constitutional Convention held, from 1961-1962, provided for four additional Senate Districts and ten additional House Districts, increasing the Senate from 34 to 38 districts and the House from 100 to 110 districts. These two changes were intended to provide additional representation for the largest districts, with the moiety clause modifying the size of the reapportionment and change in district allocation.

Given a large number of small counties, the moiety districts varied by the size of the counties and therefore any regional district allocations involving both a large area and a large number of counties. This result describes the 1951 and 1961 reapportionments and redistricting from single county representation and multi-county districts, limited in the number of counties, too regional districts covering larger areas in square miles with multi-county solutions ($MC \geq 4$ counties) and both sub-state regions and multiple urban areas.

The changes in the size of the Michigan Legislature were seemingly for the purposes of allowing for another increase in the Legislature, from 38 to 40 Senate Districts and 110 to 120 House Districts. The potential for another 2 and 10 increase appears to have been for the purposes of additional representation to the largest districts and a provision reducing the complications of regional districts allocated to large areas of the State. The moiety clause places no range bounds on the number of counties per-District. Even so, these Michigan counties tended to involve the largest areal counties in square miles.

Definition 8.0 County population share = County Population / State Population.

Theorem 11.0 Population Ratio = Population Share • Size of the Legislature.
Proof. **Definition 8.0**. $S \bullet N = d_m$. **Lemma 6.0**. $d_m = P[r]$.

Proposition 13.0 Senate Population Ratio = Population Share • Size of the State Senate.

Proposition 14.0 House Population Ratio = Population Share • Size of the State House.

Definition 9.0 County land share = County land / Total State Land.

Lemma 8.0 Moiety District Allocation = Area Factor + Population Factor.
Proof. Moiety clause ≡ Population : Area ratio. P : A = 4:1.

Lemma 9.0 Moiety Factor = (400 • Population Share) + (100 • Land Share).
Proof. AP Factor = $(400 \bullet S_p) + (100 \bullet S_L)$ = Moiety District allocation.

Lemma 10.0 Area Factor = (1 / number of local jurisdictions).
Proof. Fragmentation solution = $\mathscr{F}^*$. $\mathscr{F}^*$ = J ≡ number of local jurisdictions. 1/ J = an equal weighting of the local jurisdictions. 1 / J ⇒ an areal weight or an area factor.

Lemma 11.0 Population Factor = (1 / number of districts).
Proof. D = the number of electoral districts. 1 / D = an equal weighting of the electoral districts. 1 / D ⇒ a population weight or factor.

Proposition 15.0 Area factor (afactor) = 1 / J = 1 / number of counties.

Proposition 16.0 Senate Population factor (pfactors) = 1 / N = 1 / Senate.

Proposition 17.0 House Population factor (pfactorh) = 1 / N = 1 / House.

Proposition 18.0 Senate Factor (sfactor) = (400 • Population Factor Senate) + (100 • Area Factor).

Proposition 19.0 House Factor (hfactor) = (400 • Population Factor House) + (100 • Area Factor).

Proposition 20.0 County Area and Population Factor (apfactor) = Moiety = (400 • County Population Share) + (100 • County Land Share).

Proposition 21.0 Senate Apportionment Factor (appfactors) = County Area and Population Factor (apfactor) / Senate Factor (sfactor) ≈ Senate Population Ratio.

Proposition 22.0 House Apportionment Factor (appfactorh) = County Area and Population Factor (apfactor) / House Factor (hfactor) ≈ House Population Ratio.

Proposition 23.0 Senate Population Ratio (sratio) = County Population Share • size of the Senate (Senate).

Proposition 24.0 House Population Ratio (hratio) = County Population Share • size of the House (House).

TABLE 2.0 Descriptive Statistics on U. S. Counties from 1900 to 1990

	N	Mean	Std. Error	Std. Deviation	Skewness	Kurtosis
AFACTOR	31491	.01703	.00013	.02339	7.468	73.222
PFACTORS	31337	.02638	.00005	.00831	4.485	44.833
PFACTORH	31470	.00964	.00003	.00451	5.144	47.637
SFACTOR	31332	12.2371	.02834	5.0169	6.216	63.359
HFACTOR	31451	5.5468	.02037	3.6119	6.705	65.970
APFACTOR	30585	8.0213	.09582	16.7567	8.922	118.284
APPFACTORS	30500	.61093	.00583	1.01883	8.272	117.051
APPFACTORH	30580	1.32899	.01235	2.16036	9.575	166.306
SRATIO	30583	.60201	.00780	1.36334	9.740	149.568
HRATIO	30657	1.85213	.02956	5.17526	10.455	158.063
Valid N	30500					

TABLE 3.1 Shapiro-Wilk W test for normal data

Variable	Obs	W	V	z	Prob>z
afactor	31491	0.48390	6664.822	24.211	0.00000
pfactors	31337	0.79637	2619.368	21.641	0.00000
pfactorh	31470	0.88454	1490.294	20.092	0.00000
sfactor	31332	0.57678	5443.427	23.652	0.00000
hfactor	31451	0.50617	6370.737	24.087	0.00000
apfactor	30585	0.34730	8235.547	24.779	0.00000
appfactors	30500	0.41438	7372.792	24.473	0.00000
appfactorh	30580	0.39124	7680.160	24.587	0.00000
sratio	30583	0.33837	8347.813	24.816	0.00000
hratio	30657	0.26863	9245.509	25.098	0.00000

TABLE 3.2 Shapiro-Francia W' test for normal data

Variable	Obs	W'	V'	z	Prob>z
afactor	31491	0.48401	8505.083	25.589	0.00001
pfactors	31337	0.79636	3340.898	22.940	0.00001
pfactorh	31470	0.88471	1899.178	21.348	0.00001
sfactor	31332	0.57674	6943.015	25.008	0.00001
hfactor	31451	0.50611	8130.945	25.460	0.00001
apfactor	30585	0.34714	1.0e+04	26.131	0.00001
appfactors	30500	0.41419	9366.865	25.813	0.00001
appfactorh	30580	0.39098	9762.170	25.934	0.00001
sratio	30583	0.33824	1.1e+04	26.169	0.00001
hratio	30657	0.26847	1.2e+04	26.462	0.00001

Note: the normal approximation to the sampling distribution of W' is valid for 10<=n<=5000 under the log transformation.

TABLE 4.1 Analysis of Intra-State Variance, County Effects

Test of Homogeneity of Variances

	Levene Statistic	df1	df2	Sig.
AFACTOR	5241.228	49	31441	.000
PFACTORS	2832.486	49	31287	.000
PFACTORH	1434.736	49	31420	.000
SFACTOR	3819.649	49	31282	.000
HFACTOR	2606.072	49	31401	.000
APFACTOR	242.497	49	30535	.000
APPFACTORS	56.154	49	30450	.000
APPFACTORH	28.444	49	30530	.000
SRATIO	149.320	49	30533	.000
HRATIO	283.413	49	30607	.000

TABLE 4.2 Analysis of Inter-State Variance, State Effects

ANOVA

		Sum of Squares	df	Mean Square	F	Sig.
AFACTOR	Between States	16.170	49	.330	9749.893	.000
	Within States	1.064	31441	3.385E-05		
	Total	17.234	31490			
PFACTORS	Between States	1.728	49	3.527E-02	2545.061	.000
	Within States	.434	31287	1.386E-05		
	Total	2.162	31336			
PFACTORH	Between States	.537	49	1.095E-02	3370.206	.000
	Within States	.102	31420	3.250E-06		
	Total	.639	31469			
SFACTOR	Between States	670427.062	49	13682.185	3622.459	.000
	Within States	118153.477	31282	3.777		
	Total	788580.539	31331			
HFACTOR	Between States	366920.326	49	7488.170	5422.668	.000
	Within States	43361.689	31401	1.381		
	Total	410282.014	31450			
APFACTOR	Between States	3002455.871	49	61274.610	335.002	.000
	Within States	5585106.016	30535	182.908		
	Total	8587561.887	30584			
APPFACTORS	Between States	3715.326	49	75.823	82.626	.000
	Within States	27943.023	30450	.918		
	Total	31658.349	30499			
APPFACTORH	Between States	9675.103	49	197.451	45.311	.000
	Within States	133041.229	30530	4.358		
	Total	142716.333	30579			
SRATIO	Between States	9880.981	49	201.653	131.108	.000
	Within States	46961.839	30533	1.538		
	Total	56842.819	30582			
HRATIO	Between States	312504.366	49	6377.640	383.826	.000
	Within States	508565.103	30607	16.616		
	Total	821069.469	30656			

TABLE 5.1 Inequality measures of appfactors

relative mean deviation	.38302253
coefficient of variation	1.6676564
standard deviation of logs	.93619359
Gini coefficient	.52637926
Mehran measure	.64656108
Piesch measure	.46628837
Kakwani measure	.2329019
Theil entropy measure	.5893109
Theil mean log deviation measure	.49895127

TABLE 5.2 Inequality measures of sratio

relative mean deviation	.46861555
coefficient of variation	2.2646643
standard deviation of logs	1.216206
Gini coefficient	.62821698
Mehran measure	.74910365
Piesch measure	.56777361
Kakwani measure	.32167347
Theil entropy measure	.8790608
Theil mean log deviation measure	.77919738

TABLE 5.3 Inequality measures of appfactorh

relative mean deviation	.35893022
coefficient of variation	1.6255601
standard deviation of logs	.83531792
Gini coefficient	.49539156
Mehran measure	.60845429
Piesch measure	.43886021
Kakwani measure	.20983355
Theil entropy measure	.53633272
Theil mean log deviation measure	.42685905

TABLE 5.4 Inequality measures of hratio

relative mean deviation	.50357318
coefficient of variation	2.7942167
standard deviation of logs	1.1892741
Gini coefficient	.66308608
Mehran measure	.76904746
Piesch measure	.61010541
Kakwani measure	.36010408
Theil entropy measure	1.0782283
Theil mean log deviation measure	.85809808

Lemma 12.0	(State Apportionment) Factor = (400 • Population Factor) + (100 • Area Factor). **Proof**. D = the number of electoral districts, a finite integer set. J = the number of local jurisdictions, a finite integer set. For any individual State, the number of electoral districts and number of local jurisdictions, such as towns or counties, are constant numbers, J and D. The fixed State effect is equal to a weighted combination of the number of electoral districts and the number of local jurisdictions.
Theorem 12.0	Weighted Apportionment Factor = Moiety Factor / Apportionment Factor. **Proof**. AP Factor = $(400 \bullet S_p) + (100 \bullet S_L)$ = Moiety District allocation. Fixed State Effect = Apportionment Factor. (400 • Population Factor) + (100 • Area Factor). Apportionment Factor = Fixed Statewide effect in numbers of districts and local jurisdictions. AP Factor / Factor = Moiety Factor / Apportionment Factor.
Lemma 13.0	(Population Distribution) Log Rank Rule. **Remarks**. The log rank rule summarizes the distribution of population by local jurisdiction. This describes the structure of the population distribution by location. Log(county population) = λ•Ranking(county population).
Lemma 14.0	(Herfindahl index) Concentration of State's Population. **Proof**. County population / State Population = County Population Share. Population Share ≡ s. Σs^2 = Herfindahl index a measure of share concentration.
Proposition 25.0	(Shift-Share Analysis) Population Growth Rate = ψ(Population Share).
Theorem 13.0	(Dauer-Kelsey Ratio) DK = simple majority sum of population shares. **Proof**. Rank order the local jurisdiction (town or county) population shares in ascending order. N = size of the legislative chamber. Assuming $(N / 2) + 1 \equiv$ a simple majority and d ≡ a finite integer delegation size. Then $\Sigma d = N$. $\Sigma[(d/2) + 1]$ = simple majority of electoral districts. $\Sigma d_m \equiv$ sum of a majority of the counties population shares, in ascending order, from the smallest to the majority of the counties. The sum of the district magnitudes equals a majority county population share, $\Sigma[(d/2) + 1] \Rightarrow \Sigma[(d_m/2) + 1]$. $\Sigma[(d_m/2) + 1] = \Sigma s = .50 + 1/N$. Σs at the point of the majority of the legislative chamber is defined as the Dauer-Kelsey ratio. $\Sigma s \equiv DK$, at $\Sigma[(d_m/2) + 1] = .50 + 1/N$.

Theorem 14.0 $\mathscr{F}$(Log Rank Rule) = Dauer-Kelsey Ratio.

Proof. Log Rank Rule ≡ Log(Population) = Ranking(Population). Ranking(County Population) ≡ Ranking in descending order. Log(County Population) = Ranking(County Population). Dauer-Kelsey Ratio ⇒ Ranking(County Population) ≡ Ranking in ascending order. Dauer-Kelsey Rule = [Log Rank Rule]$^{-1}$. Dauer-Kelsey Ratio = ϕ(Apportionment) ≡ an apportionment mapping. ϕ(Apportionment) = $\mathscr{F}^*$. Setting $\mathscr{F}^*$ = $\underline{J}$, and $\underline{J}$ = λ• [Σ[(d /2) + 1]], the fragmentation solution equals a weighted majority of the district allocation. Given λ• [Σ[(d /2) + 1]] = (N / 2) + 1, the weighted majority of the districts equals a majority of size of the legislature. The ranking in ascending order is a linear, one-to-one correspondence, with the ranking in descending order. The constrained ranking in ascending order is a linear, one-to-one correspondence, with the majority percentile ranking of the cumulative population of the smallest units in the descending rank ordering. ϕ(Apportionment) = λ• [Σ[(d /2) + 1]] = (N / 2) + 1. ϕ(Apportionment) = (N / 2) + 1. ϕ(Apportionment) = Ranking(County Population) of the smallest counties = ($\underline{J}$ / 2) + 1, a majority of the local jurisdictions. A rank ordering of the population = ϕ(Apportionment).

Lemma 15.0 log rank rule = ϕ(DK).

Proof. DK = ϕ(apportionment). ϕ(apportionment) = λ•Σd = (N / 2) + 1. Ranking(county population) in descending order, largest to smallest county population. ΣP = (N / 2) + 1, from the largest to smallest ⇒ simple population majority = summation of the largest district allocations. ΣP = (N / 2) + 1, from the smallest counties to the largest ⇒ simple majority of the legislature = majority of the counties. ΣP = (N / 2) + 1, from the smallest counties to the largest ≐ DK ratio.

Theorem 15.0 log rank rule = ϕ(apportionment).

Proof. ϕ = inverse rule. log rank rule = ϕ(DK) = (DK)$^{-1}$. DK = ϕ(apportionment).

Theorem 16.0 Population share concentrations → DK ratio.

Proof. J = number of local jurisdictions (counties).

- largest county share
- 2 largest counties share
- 4-county ratio
- 6-county ratio
- 8-county ratio
- 16-county ratio

Theorem 17.0 (Classification) A structure-induced equilibrium exists in apportionment and district allocation.

- Majority of Local Jurisdictions = Majority of the Legislature.
- Fragmentation Solution = Majority of the Legislature.
- Majority of Electoral Districts = Majority of the Legislature.

Proposition 26.0 A structure-induced equilibrium (SIE) exists in a majority of districts.

Proposition 27.0 An SIE exists in delegation sizes.

Proposition 28.0 An SIE exists in district magnitude.

Proposition 29.0 An SIE exists in numbers of groups, as faction sizes in a legislature.

Proposition 30.0 An SIE exists in a majority of local jurisdictions.

Proposition 31.0 (County Representation) An SIE exists in a majority of counties.

Proposition 32.0 (Town Representation) An SIE exists in a majority of towns.

Proposition 33.0 An SIE exists that is a fragmentation solution in local jurisdiction.

Proposition 34.0 An SIE exists that is a local jurisdictionally-induced equilibrium.

Proposition 35.0 An SIE exists in fractional apportionments.

Proposition 36.0 An SIE exists in fractional district allocations.

Proposition 37.0 An SIE exists in proportionate representation permitting district rotations.

Proposition 38.0 An SIE exists in partial representation permitting fractional representation.

Proposition 39.0 An SIE exists in Flotorial or Integer Remainder Districts.

Proposition 40.0 An SIE exists in direct and indirect representation.

Proposition 41.0 An SIE exists in population ratios.

Proposition 42.0 An SIE exists in population shares.

Conjecture 1.0 Legislative apportionment to local jurisdiction is a structure-induced equilibrium.

Conjecture 2.0 Malapportionment is the degree to which minority rule is the structure-induced equilibrium.

Conjecture 3.0 Any legislative majority may be constructed from a population share, ratio equilibrium.

Theorem 18.0 A local jurisdictionally-induced equilibrium exists.
Proof. Rule(a majority or more of the local jurisdictions) = simple majority of the size of the legislature. Fragmentation solution exists and the fragmentation solution = majority of the legislature.

Theorem 19.0 (Fragmentation solution) ESS = $\underline{J} \leq$ T.
Proof. (ESS $\leq$T) $\rightarrow$ t. Duration or age of the local jurisdiction $\equiv$ current year - the organizational formation date. T - t_0 = t. A fragmentation solution is equated to the number of local jurisdictions, $\mathscr{F}^*$ = $\underline{J}$. The ESS in numbers of local jurisdictions is therefore range bound in the ages of the local jurisdictions, (j $\leq$ t) $\rightarrow$ (J $\leq$ T). Substituting $\mathscr{F}^*$ = $\underline{J}$, (J $\leq$ T) $\rightarrow$ ($\underline{J} \leq$ t).
Evidence. By 1933, 99% of the counties were formed.

Theorem 20.0 (Organizational stability) O(t) = J.
Proof. Organizational sclerosis $\equiv$ O(t). J = number of local jurisdictions, a finite integer set. J $\rightarrow$ fixed number, $\underline{J}$ = ΔJ $\approx$ 0. O(t) = $\underline{J}$ = J.

Theorem 21.0 Number of Townships = County Land Area / 36.
Proof. Define a regular Northwest (Ordinance) Township = a 6 • 6 square. Township Area = 36 square miles.

Theorem 22.0 (Compactness) Number of Townships / 36.
Proof. Given State distributions of county land area in square miles. County land area / 36 = number of townships. Dividing the number of townships / 36 = measure of the regular square shape in size of the county area in square miles. This measure equals compactness of the 37.70 area of an exterior circle, and the 32.86 area of the interior circle.

TABLE 6.0 Seemingly Unrelated Regression Analysis, 1900 - 1990 Data

. sureg

Equation 1.0	(landarea = trend duration), Census Trend (1900-1990), Age of County
Equation 2.0	(compactness = logarea trend duration), Compactness, number of Townships, Census Trend (1900-1990), Age of County
Equation 3.0	(house = county senate trend), House Apportionment
Equation 4.0	(senate = county house trend), Senate Apportionment
Equation 5.0	(hsech = house county senate trend), House Reapportionment
Equation 6.0	(sench = senate county house trend), Senate Reapportionment
Equation 7.0	(appfactorh = hratio), House Area Factor, Moiety & Population Ratios
Equation 8.0	(appfactors = sratio), Senate Area Factor, Moiety & Population Ratios
Equation 9.0	(logpop = poprank), Log Rank Rule
Equation 10.1	(growth = share concentrationratio landshare trend duration)*
Equation 10.2	(growth = log(pop) log(area) trend duration)

*shift-share analysis significant but not reported..

Seemingly unrelated regression

Equation	Obs	Parms	RMSE	"R-sq"	chi2	P
landarea	27139	2	1733.572	0.0006	112.56	0.0000
compact	27139	3	1.151213	0.2598	8265.26	0.0000
house	27139	3	36.72252	0.1969	18889.37	0.0000
senate	27139	3	9.416638	0.0693	15831.28	0.0000
hsech	27139	4	7.470979	0.0879	2648.38	0.0000
sench	27139	4	2.321834	0.0241	603.17	0.0000
appfactorh	27139	1	1.414102	0.5612	45650.57	0.0000
appfactors	27139	1	.3187317	0.8979	288347.41	0.0000
logpop	27139	1	.9124786	0.3717	18176.60	0.0000
growth	27139	4	1.289307	0.0167	701.94	0.0000

	Coef.	Std. Err.	z	P>|z|	[95% Conf.Interval]	
landarea						
trend	39.413	3.973	9.92	0.000	31.626	47.201
duration	-0.860	0.211	-4.08	0.001	-1.274	-0.447
_cons	1004.21	40.961	24.52	0.000	923.926	1084.489
compactness						
logarea	0.425	0.005	90.44	0.000	0.416	0.434
trend	0.017	0.002	6.97	0.000	0.012	0.022
duration	-0.000	0.000	-3.60	0.001	-0.001	-0.000
_cons	-1.980	0.040	-49.03	0.000	-2.059	-1.901
house						
county	0.222	0.003	63.74	0.000	0.215	0.228
senate	2.260	0.018	123.77	0.000	2.225	2.296
trend	-1.366	0.074	-18.42	0.000	-1.511	-1.221
_cons	11.840	0.875	13.54	0.000	10.126	13.554
senate						
county	-0.046	0.001	-45.69	0.000	-0.048	-0.044
house	0.161	0.001	123.96	0.000	0.159	0.164
trend	0.383	0.021	18.17	0.000	0.342	0.425
_cons	24.194	0.181	133.40	0.000	23.839	24.550
hsech						
house	-0.030	0.001	-23.86	0.000	-0.033	-0.028
county	0.011	0.001	12.93	0.000	0.010	0.013
senate	-0.067	0.005	-13.31	0.000	-0.077	-0.057
trend	-0.647	0.018	-36.26	0.000	-0.682	-0.612
_cons	8.388	0.222	37.79	0.000	7.953	8.823
sench						
house	0.002	0.000	4.82	0.000	0.001	0.003
county	-0.004	0.000	-15.07	0.000	-0.005	-0.004
senate	-0.025	0.002	-16.15	0.000	-0.028	-0.022
trend	-0.057	0.006	-10.36	0.000	-0.068	-0.047
_cons	1.869	0.069	27.17	0.000	1.734	2.004

appfactorh						
hratio	0.278	0.001	213.66	0.000	0.275	0.280
_cons	0.812	0.009	91.67	0.000	0.795	0.830
appfactors						
sratio	0.635	0.001	536.98	0.000	0.633	0.638
_cons	0.227	0.002	115.43	0.000	0.223	0.231
logpop						
poprank	-0.015	0.000	-134.82	0.000	-0.016	-0.015
_cons	10.607	0.008	1353.09	0.000	10.592	10.622
growth						
logpop	-0.165	0.007	-24.14	0.000	-0.179	-0.152
logarea	0.039	0.010	4.09	0.000	0.020	0.058
trend	-0.020	0.003	-6.51	0.000	-0.026	-0.014
duration	-0.000	0.000	-0.75	0.466	-0.000	0.000
_cons	1.606	0.099	16.24	0.000	1.422	1.800

Evolutionary Stable Strategies in Apportionment and District Plans

The failure of the largest districts to attain population ratios evolved during the 1900 to 1990 period as single county representation plans failed in numbers of districts and apportionment in delegation sizes. The failure of the largest districts is important both in terms of constructing apportionment solutions and any spatial competition amongst urban areas. As the findings indicate, the location of population ratios influences apportionment and district allocation. The distribution of population ratios determines not only the largest districts, but also the location of apportionment and district allocation. Each Western State, for example, has a unique distribution of population ratios and therefore locations of core and periphery areas, duopoly cores and duopoly periphery area, and more generally, numbers of large counties and most populated districts. As the results on Senate Apportionment reveal, the potential for largest district solutions varied in all of The States.

The largest district solutions may be described as large multi-member districts that may have additional voting rules and procedures to reduce the combination of seats or positions or the number of voting alternatives. As reported in **Appendix II**, the Senate Apportionments ranging from 1.5 to 2.75 population ratios generated a range of choices to allocate two districts across The States. These findings also suggest variation in Senate Apportionment and District allocation plans for the largest districts by choice of multi-member districts, county subdivision, and mixed single county representation plans with city and county districts. The evolution of the largest district solutions and district allocation plans varied in population ratios, location effects, delegation sizes, apportionment, district plans, local division, and numbers of local jurisdictions.

As single county representation plans evolve, regional districts have been constructed from combinations of 6 to 15 or more counties in The States. These regional districts describe location effects on apportionment and district allocation plans. In the Western States, these may be either located by sub-state regions or relative to the spatial distribution of the largest districts. These regional districts are determined by the location of small and marginal counties that require some multi-county solution in what has become, a large number of counties per-District. As a consequence, the largest district solutions imply multi-county solutions in the form of regional districts. These districts may require combinations of a large number of contiguous counties, in a single sub-state region, or connections of peripheral areas from two or more metropolitan areas.

The evolutionary stable strategy is different for Senate and House Apportionment. From 1881 to 1991, the County Senate District Plan evolved from single to multi-county solutions, averaging between 2 to 4 counties per-district. This multi-county solution may be described in terms of delegation size, $d = \{2, 3, 4\} = j$, in numbers of counties and districts allocated. For many States, the evolution of a County Senate Plan implied a Senate Apportionment and District allocation plan in two or three counties. As the largest districts increased in population ratios, spatial competition produced an increase in the number of counties per-district and allowed for variable numbers of counties to be combined. By implication, this produced an increase in the importance of contiguous formation of multi-county districts, and a reduction in the compactness of these districts as the number of counties combined increases. The failure of single county representation produces therefore an increasing reliance on multi-county solutions and an increasing complexity in the location and number of counties combined.

The changes in the distribution of population ratios resulted in the imposition of constraints for counties to attain population ratios. For Ohio, the 1907 Hanna Amendment guaranteed at least one House District per-county. The findings indicate the average population ratio, from 1900 to 1990, equals 1.85. This result produces a range bound of delegation size, d = {1, 2} = SMD + MMD (= 2), in General Apportionment and District allocation plans.

In summary, the evolutionary stable strategy varies in Senate and House Apportionment. The ESS for the Senate Apportionment and District (allocation) Plan equals, SMD = 1, MC = 2, providing for single member Senate Districts consisting of two counties. The ESS for House Apportionment and District (allocation) Plan equals, MMD = 2, SC = 1, that provides for two districts for each county. These basic results describe the general legislative apportionment to local jurisdiction by district allocation plans. These evolutionary stable strategies (ESS) are derived from changes in the distribution of population ratios for a ninety-year duration, for a spatial history of Apportionment and District allocation Plans. As shown by the results, this ESS changes by population ratios in apportionment and district plans, including those designed for single county representation plans. The Senate (1, 2) and House (2, 1) Plans evolved during a shorter period of changes in the distribution of population ratios, than other legislatures, such as the English county and borough apportionment of 2-members for each electoral district. Among The States, multiples (or MMD = 2) have been used for Senate Apportionment in Arizona, where the 1953 Pyle Amendment guaranteed 2 Senate Districts per-county to adjust for House Apportionment based on votes in gubernatorial elections. Other states have adopted provisions requiring House Apportionment to be contained within Senate Districts, for a fixed size of the legislature and usually a fixed (2:1 or 3:1) ratio in numbers of House to Senate Districts.

FIGURE 2.0

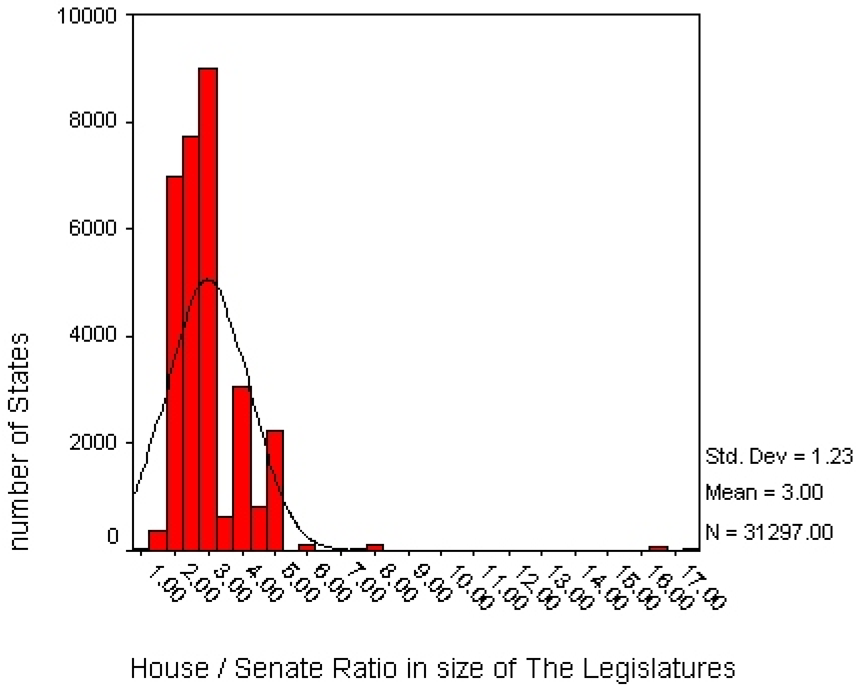


TABLE 7.0 Descriptive Statistics on Bicameral Ratio in sizes of The Legislatures

	N	Mean	Std. Error	Std. Deviation	Skewness	Kurtosis
HSRATIO	31297	2.9977	.00696	1.2311	5.138	50.184

In settings with areal containment requirements for district allocation ($H \subseteq S$), The States tend to fix the bicameral ratio in the relative sizes of the House to Senate seats or positions. In States or Territories with Island-County representation, the House Districts are not contained within Senate Districts. Apportionment and district allocation evolves by plan and legislative chamber. In Hawaii, one of the four counties, the least populated county of Kauai, attained population ratios equal to 0.962 (Senate) and 1.963 (House) in 1970 and 1.013 (Senate) and 2.066 (House) in 1980. In Puerto Rico, a County Senate District Plan was enacted in 1917, consisting of 7 Senate Districts that were expanded to 8 Districts in 1952. The mixed representation plan adopted has evolved to a 21.6% (House) and 40.7% (Senate) MMD Plan.

TABLE 8.0 Legislative Apportionment in Puerto Rico by Year of provision, size of the legislature, Group Decision Function or Range-Density Solution, number of Local Jurisdictions (Municipalities), and percentage of the Legislative Chamber in Multi-Member Districts

House District Plan				
1917	39	(5) 7 + (4) 1	76	10.2
1952	51	(5) 8 + (11) 1	78	21.6
1964	51	(5) 8 + (11) 1	78	21.6
County Senate Plan				
1917	19	(2) 7 + (5) 1	76	26.3
1952	27	(2) 8 + (11) 1	78	40.7
1964	27	(2) 8 + (11) 1	78	40.7

TABLE 9.0 Apportionment and District Allocation in the Virgin Islands

Virgin Islands (Saint Croix, Saint Thomas, Saint John)

3 Commissioners appointed (prior to 1917)

3 Representatives of the Island City Councils (1917)

Apportionment (1, 1, 1) from each Island

3 Elected, At-Large

(2, 2, 2) and (3, 3, 3) for single elections

9 Elected, At-Large

9 Districts recommended

1954 Apportionment, (5, 5, 1) = 11 = (5) 2 + (1) 1

1966 Apportionment, (7, 7, 1) = 15 = (7) 2 + (1) 1 = 1•SMD + 2•MMD

2 multi-member districts = Saint Croix and Saint Thomas & Saint John

1 single member district = Saint John, voted on At-Large, residency requirement

Legislative apportionment is complicated by the relatively equal population sizes in Saint Croix and Saint Thomas. Of the three Saint John is the least populated, so that the Senate Apportionment is for 2 multi-member districts, with 7 representatives elected from each Island District. The “fifteenth” Senator is elected At-Large, but is required to have residency in Saint John to guarantee at least one representative per-Island. Because Saint Thomas has approximately 92-95% of the population in Saint Croix, 1) Saint John would not receive representation without a residency requirement, and, 2) the size of the unicameral Legislature would have to be increased to guarantee Saint John a single member district.

The numbers of States with SMD, MMD, and mixed representation plans reveals the choice between SMD, MMD, and SMD and MMD plans. The results in **TABLE 10.1** describe an evolving sequence of district plan choices from 1900 to 1990. The time series in **TABLE 10.1** indicates 1967 is the critical point in changes in district allocation. Inasmuch these results revealed steady decline in the use of multi-member district plans and mixed, Senate and House, representation plans having one legislative chamber elected from SMDs and the other with some proportion of MMDs. The transition to MMDs increased from 1917 throughout the 1920's, whereas the decline in MMDs increased in 1966-1972 and then continued through 1990.

As shown in **TABLE 10.2**, the inclusion of Porto Rico, Virgin Islands, and territorial data for Alaska, Hawaii, Arizona, and New Mexico, demonstrates that MMD and mixed representation plans have been used frequently by The States and Territories for the purposes of legislative apportionment. The effort to impose SMD Plans only implies the number of electoral districts equals the size of the legislative chamber, D = N. As the choice of district plans converges to an SMD Plan, the number of MMDs & mixed representation plans decreases.

TABLE 10.1 Numbers of States* with MMD Plans by Year

YEAR	SENATE	HOUSE	
1900	20	42	
1901	20	42	
1902	20	42	
1903	19	42	New Mexico-Senate-SMD
1904	19	42	
1905	20	42	New Mexico-Senate-MMD
1906	20	42	
1907	21	43	Oklahoma Senate & House MMD
1908	21	43	
1909	21	43	
1910	20	43	New Mexico-Senate-SMD
1911	21	43	Alaska-Senate & House-MMD
1912	21	43	
1913	21	43	
1914	21	43	
1915	21	43	
1916	21	43	
1917	23	44	Virgin Islands-Senate-MMD, Porto Rico-Senate-MMD, Porto Rico-House-MMD
1918	23	44	
1919	23	44	
1920	23	44	
1921	23	44	
1922	23	44	
1923	23	44	
1924	23	44	
1925	22	44	Florida-Senate-SMD
1926	22	44	
1927	22	44	
1928	22	44	
1929	22	44	
1930	22	44	
1931	22	44	
1932	22	44	
1933	22	44	
1934	22	44	
1935	22	44	
1936	22	44	
1937	22	43	Nebraska-House
1938	22	43	
1939	22	43	
1940	22	43	
1941	22	43	
1942	22	43	
1943	22	43	
1944	22	43	
1945	22	43	
1946	22	43	
1947	22	43	
1948	22	43	
1949	22	43	
1950	22	43	
1951	22	43	
1952	22	43	
1953	22	43	
1954	22	43	
1955	22	43	
1956	22	43	

1957	22	43	
1958	22	43	
1959	22	43	
1960	22	43	
1961	22	43	
1962	22	43	
1963	22	43	
1964	22	43	
1965	20	41	Oklahoma & Utah-Senate & House-SMD
1966	22	42	North Dakota-Senate-MMD, South Carolina-Senate-MMD Vermont-House-MMD
1967	18	32	IN-S-SMD, MT-S-SMD, OH-S-SMD, WA-S-SMD, AL-H-SMD, AR-H-SMD, CT-H-SMD, MI-H-SMD, MO-H-SMD, NM-H-SMD, OH-H-SMD, PA-H-SMD, SC-H-SMD, TN-H-SMD
1968	17	32	VA-S-SMD
1969	16	30	ME-S-SMD, IA-H-SMD, ME-H-SMD
1970	16	30	
1971	15	30	AZ-S-SMD
1972	15	30	
1973	11	22	CO-S-SMD, LA-S-SMD, NV-S-SMD, OR-S-SMD CO-H-SMD, FL-H-SMD, KS-H-SMD, LA-H-SMD, MA-H-SMD, MT-H-SMD, NV-H-SMD, OR-H-SMD
1974	11	21	KY-H-SMD
1975	11	21	
1976	11	21	
1977	10	20	ND-S-SMD, TX-H-SMD
1978	10	20	
1979	10	20	
1980	10	20	
1981	10	20	
1982	9	19	HI-S-SMD, HI-H-SMD
1983	8	18	SD-S-SMD, IL-H-SMD
1984	7	16	MS-S-SMD, MS-H-SMD, VA-H-SMD
1985	7	15	ID-H-SMD
1986	7	15	
1987	7	15	
1988	7	15	
1989	7	15	
1990	7	15	
Total	1710	3403	

*Number of States includes, Hawaii 1900-1990, Alaska 1913-1990, Porto Rico 1917-1990, and Virgin Islands 1917-1990.

TABLE 10.2 State by Apportionment and District allocation Plan

	PLAN			Total
STATE	SMD	MIXTURE	MMD	
AL	24	67		91
	26.4%	73.6%		100.0%
AK			80	80
			100.0%	100.0%
AZ		20	71	91
		22.0%	78.0%	100.0%
AR	24	67		91
	26.4%	73.6%		100.0%
CA	91			91
	100.0%			100.0%
CO	18		73	91
	19.8%		80.2%	100.0%
CT	24	67		91
	26.4%	73.6%		100.0%
DE	91			91
	100.0%			100.0%
FL	18	48	25	91
	19.8%	52.7%	27.5%	100.0%
GA		91		91
		100.0%		100.0%
HI	9		82	91
	9.9%		90.1%	100.0%
ID	6	85		91
	6.6%	93.4%		100.0%
IL	8	83		91
	8.8%	91.2%		100.0%
IN		24	67	91
		26.4%	73.6%	100.0%
IA	22	69		91
	24.2%	75.8%		100.0%
KS	18	73		91
	19.8%	80.2%		100.0%
KY	17	74		91
	18.7%	81.3%		100.0%
LA	18		73	91
	19.8%		80.2%	100.0%
ME	22		69	91
	24.2%		75.8%	100.0%
MD		91		91
		100.0%		100.0%
MA	18	73		91
	19.8%	80.2%		100.0%
MI	24	67		91
	26.4%	73.6%		100.0%
MN	91			91
	100.0%			100.0%
MS	7		84	91
	7.7%		92.3%	100.0%
MO	24	67		91
	26.4%	73.6%		100.0%
MT	18	6	67	91
	19.8%	6.6%	73.6%	100.0%
NE	54	37		91
	59.3%	40.7%		100.0%
NV	18		73	91
	19.8%		80.2%	100.0%

NH		91		91
		100.0%		100.0%
NJ		91		91
		100.0%		100.0%
NM	24	59	8	91
	26.4%	64.8%	8.8%	100.0%
NY	91			91
	100.0%			100.0%
NC		91		91
		100.0%		100.0%
ND		80	11	91
		87.9%	12.1%	100.0%
OH	24		67	91
	26.4%		73.6%	100.0%
OK	26		58	84
	31.0%		69.0%	100.0%
OR	18		73	91
	19.8%		80.2%	100.0%
PA	24	67		91
	26.4%	73.6%		100.0%
RI	80	11		91
	87.9%	12.1%		100.0%
SC		90	1	91
		98.9%	1.1%	100.0%
SD		8	83	91
		8.8%	91.2%	100.0%
TN	24	67		91
	26.4%	73.6%		100.0%
TX	14	77		91
	15.4%	84.6%		100.0%
UT	26		65	91
	28.6%		71.4%	100.0%
VT		66	25	91
		72.5%	27.5%	100.0%
VA	7	16	68	91
	7.7%	17.6%	74.7%	100.0%
WA		24	67	91
		26.4%	73.6%	100.0%
WV			91	91
			100.0%	100.0%
WI	91			91
	100.0%			100.0%
WY			91	91
			100.0%	100.0%
PR			74	74
			100.0%	100.0%
VI			74	74
			100.0%	100.0%
Total	1113	1947	1620	4680
	23.8%	41.6%	34.6%	100.0%

The findings in **TABLE 10.2** reveal State choices in District Allocation Plans. The trend indicates the decline in multi-member districts, for the apportionment and district plans enacted during the 20th Century. Even so, the evidence suggests The States and Territories frequently adopted mixed representation plans, with one legislative chamber elected by single member districts and the other with some multi-member districts. As shown in **TABLE 1.0, 10.1 & 10.2**, most of the State adjustments are from 1965 to 1973, with 1967 and 1973 the two most active years for new choices in District Allocation Plans.

The trend in reapportionment is reported in **TABLE 10.3**. In this model, the changes in the size of The Legislatures are estimated by the sequence of censuses, for the House and Senate Apportionment Plans. These findings reveal a significant decreasing trend in the size of the reapportionments, from 1900 to 1990. The four most significant individual reapportionments involve large numbers of House Districts in Connecticut, Illinois, Massachusetts, and Vermont. These four reapportionments produce the most significant changes in the sizes of The Legislatures. Other estimates reveal no significant differences between changes in the Senate and House apportionments. These findings indicate a trend estimate equal to a 1.62 decline, or one to two seats & position decrease that describes the ESS in the sizes of The State Legislatures. The findings in **TABLES 10.1 & 10.2** suggest the ESS in Apportionment and District allocation Plans is converging toward SMD Plans marginally, in numbers of reductions in the sizes of The Legislatures. This two dimensional strategy implies the choice of SMD Plans in one dimension, with continuously smaller numbers of seats or positions to apportion in another dimension. The ESS in this pursuit of changes in Apportionment and District allocation, implies fewer seats or positions to provide for an increasing amount of competition.

TABLE 10.3 Evolution of Reapportionment in changes in the size of The Legislatures

Model Summary

Model	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	.271	.074	.070	14.92	1.970

a Predictors: (Constant), TREND (linear sequence in decennial Census)
b Dependent Variable: REAPPORTIONMENT (House & Senate changes in the size of The Legislatures)

ANOVA

Model		Sum of Squares	df	Mean Square	F	Sig.
1	Regression	4167.195	1	4167.195	18.730	.000
	Residual	52508.099	236	222.492		
	Total	56675.294	237			

a Predictors: (Constant), TREND
b Dependent Variable: REAPPORTIONMENT

Coefficients

		Unstandardized Coefficients		Standardized Coefficients	t	Sig.	95% Confidence Interval for B	
Model		B	Std. Error	Beta			Lower Bound	Upper Bound
1	(Constant)	8.210	1.641		5.002	.000	4.977	11.443
	TREND	-1.620	.374	-.271	-4.328	.000	-2.358	-.883

a Dependent Variable: REAPPORTIONMENT

Casewise Diagnostics

Case Number	Std. Residual	REAPPORTIONMENT	State Legislative Chamber
103	-7.742	-117	Connecticut House
124	-6.335	-96	Vermont House
131	-5.153	-80	Massachusetts House
141	-3.637	-59	Illinois House

a Dependent Variable: REAPPORTIONMENT

Changes in Structures of Legislative Apportionment & District Allocation

Theorem 23.0 A structure-induced equilibrium (SIE) exists by single county representation plan.

Proof. J = number of counties. (J / 2) + 1 = majority of counties. N = size of the legislature. (N / 2) + 1 = majority of the legislature. Single county representation plan = number of local jurisdictions (J). $\Sigma SC = J$, the number of local jurisdictions equal the number of counties. D = number of districts. An SC plan allocates $\Sigma SC = D$. $\Sigma SC = D \Leftrightarrow SC = D$. $D = SC \equiv$ county districts. County districts $\Rightarrow$ single county representation plan. $\Sigma SC = D = N \equiv$ single county representation plan. N = J, AR = 0. (N / 2) + 1 = (J / 2) + 1. $(\Sigma SC / 2) + 1$ = simple majority.

Theorem 24.0 A structure-induced equilibrium (SIE) exists by single town representation plan.

Theorem 25.0 A local jurisdictional-induced equilibrium exists equal to fragmentation solution.

Proof. $\mathscr{F}^* = J$. D = J. J = N. (J / 2) + 1 = (N / 2) + 1. $(N / 2) + 1 = \mathscr{F}^*$.

Theorem 26.0 (Local Districts) A modified local jurisdiction representation plan converges to single local jurisdictional district allocation.

Proof. A modified local jurisdiction representation plan $\equiv \lambda \bullet \Sigma J = D$. $\lambda \bullet D = N$. $\lambda \bullet \Sigma J = N$. $\lambda \bullet \Sigma J = \lambda \bullet D$. $(\lambda \bullet \Sigma J = \lambda \bullet D) \rightarrow N = J$.

Theorem 27.0 (Size of the legislature) $N = \lambda \bullet J$.

Proof. $\lambda = [.5, 1]$. N = J. $N = \frac{1}{2} \bullet J$.

Remarks. J = #D = N, N = J, size of the legislative chamber = number of counties, fragmentation solution. N = #D = J, N = MC = 2, size of the legislative chamber = ½ the number of counties. MC = 3 or MC = $\sigma = \{1, 2, 3\}$ generate apportionment and district allocations derived from modifications of the number of local jurisdictions.

Theorem 28.0 A local jurisdictional-induced equilibrium exists in size of The Legislatures.

Proof. J = number of local jurisdictions. N = size of the legislature. Local jurisdiction margin ≡ J / N. J / N = λ= [0, 2].

Verification. New Hampshire = .02. Texas = 1.90.

FIGURE3.0

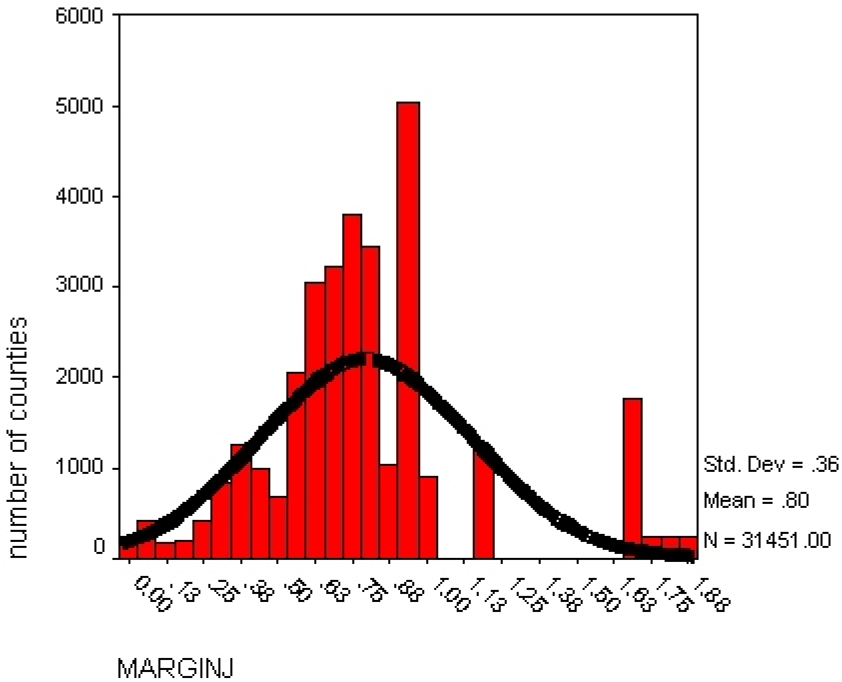


TABLE 11.0 Descriptive Statistics, Local jurisdictional margin in sizes of The Legislatures

N	Minimum	Maximum	Mean	Std. Error	Std. Deviation	Skewness	Kurtosis
31451	.02364	1.89063	.79677	.00201	.35718	1.043	1.749

Theorem 29.0 (Senate Apportionment I) An SIE exists with no more than one district allocation.

Proof. Population share = local jurisdiction population / State population (in a decennial Census). Then set population ratios = population share • size of the legislature. For population ratios greater than one, provide for a single district allocation, $d \leq 1$, with $d < 1 \Rightarrow$ MC districts and d = 1 a SC district allocation. Population ratio $\geq 1 \Rightarrow d \leq 1$.

Verification. Appendix II, **TABLE 1.0** describes counties and parishes with Senate population ratios equal to [1.493, 28.377] in the 1900 Census.

Theorem 30.0 (Senate Apportionment II) An SIE exists guaranteeing at least one district allocation.

Proof. Population share = local jurisdiction population / State population (in a decennial Census). Then set population ratios = population share • size of the legislature. For population ratios greater than one, provide for at least one district allocation, $d \geq 1$, with d = 1 a SC district allocation and d > 1 an MMD allocation. Population ratio $\geq 1 \Rightarrow d \geq 1$.

Verification. Appendix II, **TABLE 3.0** describes counties and parishes with Senate population ratios range bound between [.943, 1.483].

Proposition 43.0 $\sigma = \{1\}$, N = J, county or town district plan.

Proposition 44.0 $\sigma = \{1\}$, $N = \lambda \bullet J$ = single town or county representation plan, modified local jurisdictional plan, SC + λ•MC, ST + λ•MT district allocation.

Proposition 45.0 $\sigma = \{1, 2\}$, SC + MC district plan.

Proposition 46.0 $\sigma = \{1, 2, 3\}$, SC + MC, population classification of county districts.

Proposition 47.0 $\sigma = \{1, 2, 3, 4\}$, range-bound integer classification, large district allocations.

Proposition 48.0 $\sigma \geq 4$, d = 1, regional district.

Theorem 31.0 Single and Multi-County District Plan, with a fixed number of counties per-District.

Proof. SC + MC = D. D = {1, ..., m}. J = {1, ..., j}. $\underline{\sigma} \equiv$ greatest least upper range bound. $J = \mathscr{F}^*$, a jurisdictional fragmentation solution. $\mathscr{F}^* = \underline{\sigma}$. For any D = 1, $\underline{\sigma} = j$.

Theorem 32.0 Single and Multi-County District Plan, with a range bound number of counties per-District.

Proof. Theorem. $\underline{\sigma} = [1, j]$.

Theorem 33.0 Single and Multi-County District Plan, with a minimum number of counties per-District.
Proof. At least one county per-district, $j \geq 1$. $\underline{\sigma} \geq 1$, for any $D = 1$.

Theorem 34.0 Single County Representation Plan and County District Plan, with a maximum number of counties per-District.
Proof. No more than one county per-district, $j \leq 1$. $\underline{\sigma} = 1$, for any $D = 1$.

Theorem 35.0 Single and Multi-County District Plan, with a maximum number of counties per-District.
Proof. No more than a maximum delimitation, $\underline{\sigma} \equiv$ greatest least upper range bound. For any $D = 1$, $j \leq \underline{\sigma}$.

Theorem 36.0 Single and Multi-County Representation Plan.
Proof. $SC + MC = D$. $J = \{1, ..., j\}$. Range-bound $\equiv \sigma$. $\sigma = [1 \leq \underline{\sigma} \leq j]$.

Theorem 37.0 Single and Multi-County District Plan, with a range bound number of counties per-District and a single regional district with a large number of counties.
Proof. $SC + MC = D$. $J = \{1, ..., j\}$. Range-bound $\equiv \sigma$. $\sigma = [1 \leq j \leq \underline{\sigma}]$.

Theorem 38.0 Single and Multi-County District Plan, $SC + MC = N$.
Proof. $SC + MC = \lambda \bullet J$. $\lambda \bullet J = D$. $D = N$.

Theorem 39.0 (No more than a single district allocation) Population ratio ≥ 1, $D = 1$.

Theorem 40.0 (At least one district allocation) $J = 1$, $D = 1$.

Theorem 41.0 At least one district allocation per-County $\doteq$ no allocation of county division districts.
Proof. $D \geq 1$, $J = 1$. $J = \lambda \bullet SC$. $\lambda \bullet SC = D$. $D = N$. Assume $MC = 2$. $\lambda \bullet MC = \mathscr{F}^*$, a fractional representation plan such as county division districts. $J(B) \equiv$ local jurisdictional boundaries. $J = MC = 2$. $J = \{1, 2\}$. Define an area function $\ell(j) = J(B)$. Setting $d = \lambda \bullet B^2 = \phi(D)$, in delegation size, district boundaries, and number of districts. $\mathscr{F}^* = \lambda \bullet MC = \lambda \bullet SC(1) = \lambda \bullet SC(2)$. $\lambda \bullet SC(1) = \lambda \bullet SC(2) = (\ell(1)/\lambda_1) + (\ell(2)/\lambda_2) = J(B_1) + J(B_2) \equiv$ local jurisdiction division district. For any $D = 1$, $\lambda \bullet SC = \ell(j) = J(B) = d = 1 = \lambda \bullet B^2 \equiv$ legislative apportionment to fractions of local jurisdictions. Assuming $D \geq 1$, $J = 1$. $J = \Sigma SC$. $\lambda \bullet \Sigma SC = D$. $D = N$. $\lambda \bullet SC(1) \neq \lambda \bullet SC(2) \neq \lambda \bullet MC \neq \mathscr{F}^*$.

Theorem 42.0 A fragmentation solution exists in constrained $\lambda \bullet MC = 2$ solutions.

Proof. Define a Flotorial or partial remainder District $\equiv$ $F(D) = I = \{1, ..., m\}$. $J = \{1, ..., j\}$. $F(D) = \lambda \bullet J = p = [0, \lambda \bullet j] = J + AR = N$. $N = J + F(D)$. $J + F(D) = \lambda \bullet J$. $\lambda \bullet J = \mathscr{F}^*$.

Theorem 43.0 Classification of Legislative Apportionment to Local Jurisdiction.

- County Senate District Plan
- AL-D, ESS D = J
- D, MMD or CSD (county subdivision districts)
- AL-R, at least one seat-position guaranteed in the Legislature
- J, N, D, population ratio apportionment and district allocation
- fixed N, zero-sum apportionment and district allocation
- county subdivision districts
- min, max delegation size and size of the legislature
- flotorial district allocation, district magnitude and allocation size
- no more than one district allocation per-local jurisdiction
- federal plan guaranteeing area and population factors
- unicameral plan, Senate District Plan
- Senate District Plan, $H \subseteq S$ with a fixed bicameral ratio district allocation
- proportionate representation
- partial representation
- frozen or permanent district plans congruent with local division
- AL-D, AL-R, county allotment, subdistricting, CSD
- AL-D, AL-group plan, grouped on ballot seat-positions
- AL-D, AL-place, residency requirement
- county subdivision districts, CSD
- CSD + SMD = D
- CSD + MMD = D
- mixed representation plans = SMD + MMD
- MMD $\rightarrow$ SMD
- county division districts

Theorem 44.0 Classification of Apportionment and District Allocation, finite integer solutions in Delegation Sizes

- $\sigma = \{1\}$, N = J, county or town district plan
- $\sigma = \{1\}$, $N = \lambda \bullet J$ = single town or county representation plan, modified local jurisdictional plan, SC + MC, ST + MT district allocation
- $\sigma = \{1, 2\}$, Connecticut town solution
- $\sigma = \{1, 2, 3\}$, Georgia county solution
- $\sigma = \{1, 2, 3, 4\}$, variable delegation sizes (Alabama county solution)
- $\sigma = \{1, 2, 3, 4, 5\}$, Maryland county solution
- $\sigma = \{1, 2, 3, 4, 5, 6\}$, New Jersey county solution
- $\sigma = \{1, 2, 3, 4, 5, 6, 7\}$, Missouri county solution
- $\sigma = \{1, 2, 3, 4, 5, 6, 7, 8\}$, Alabama redistribution from the largest district solution.

Theorem 45.0 Territorial integrity, local jurisdictional classification

- no county division districts
- no county division
- county boundaries intact
- subdistricting

Proof.

☐ $d < 1$
☐ $d = 1$
☐ $d = 2$
☐ $d > 2$
☐ $d = \{3, 4\}$.

Theorem 46.0 At-Large Election guarantees at least one district allocation.

☐ Additional Representation, F(D), Flotorial District, MC solution
☐ SMD
☐ MMD
☐ At-Large group ballot: paired comparisons by number, seat or position
☐ At-Large place by residency areas, zonal districts

Theorem 47.0 County Subdivision District Plan

- County subdivision ≠ SC, SMD.
- County subdivision ≠ SC, MMD, At-Large election.
- County subdivision = SC, 2•SMD = 1, SMD subdistricting.
- County subdivision = SC, 2•MMD = 2, MMD subdistricting.

Proof. Local division ≡ town, township, city, city ward, town sections, village. Subdivision ≡ finer or coarser partition ≡ ϕ. County subdivision = partition congruent with local division by (N, S), NW, NE, SW, SE and/or (E, W), E, W/C, EC, WC. In two dimensions, a 4x4 partition in location and distance by subdivision or local (Midland) division.

Theorem 48.0 County Allotment by Population Ratio Allocation

- County subdivision = SC, SMD.
- County subdivision = SC, MMD, At-Large election.
- County subdivision = SC, 2•SMD = 1, SMD subdistricting.
- County subdivision = SC, 2•MMD = 2, MMD subdistricting.

Proof. County allotment ⇒ either a single member district or multi-member district plan for at-large election. County allotment ⇒ subdivision. County subdivision ⇒ either a single member district or multi-member district plan. Subdivision ⇒ subdistricting. Local division ⊆ district allocation plan. District Allocation Plan = Population Ratio Allocation = County Allotment.

Theorem 49.0 County Subdivision Game

- status quo, SMD, MC = 2
- SC, SMD
- SC, AL-D, 1•MMD = 2
- SC, CSD, 2•SMD = 1, ESS.

Theorem 50.0 Multi-County Division Game

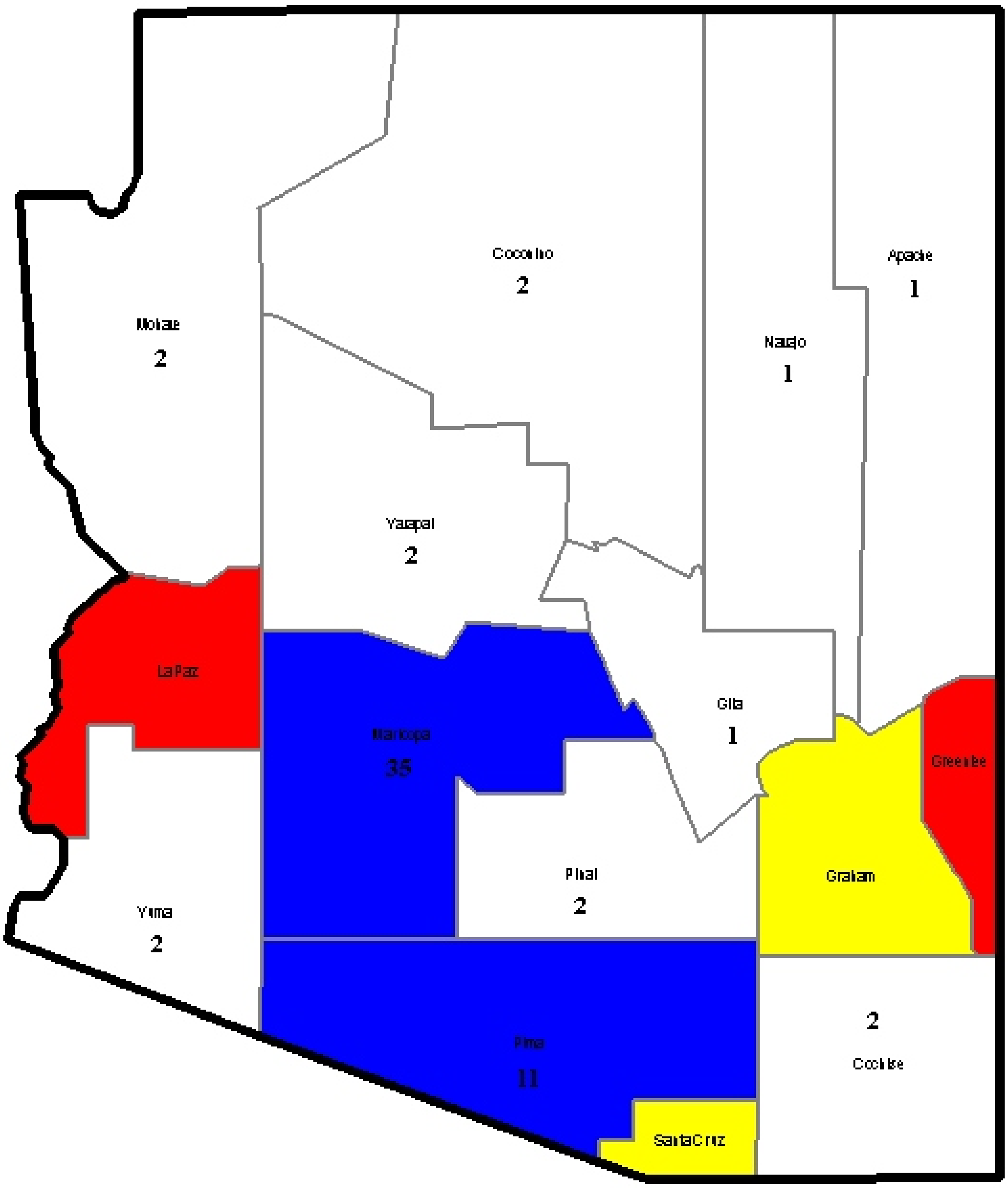
- status quo, SC, county representation plan
- SC + MC, multi-county solutions for district allocation
- SC + (MC = 2)
- SC (MMD) + (MC = 2)
- SC (MMD) + (MC = 3)
- SC (CSD, 2•SMD = 1) + (MC = 3)
- SC (CSD, λ•SMD = 1) + (MC = 4, Regional District)
- SC (CSD, λ•SMD = 1) + CDD(λ•MC = 2) + (MC = 4, Regional District)
- SC (CSD, λ•SMD = 1) + MC(CDD, λ•SMD = 1)
- CSD + CDD, county subdivision + county division = apportionment
- single county subdivision + multi-county division, ESS.

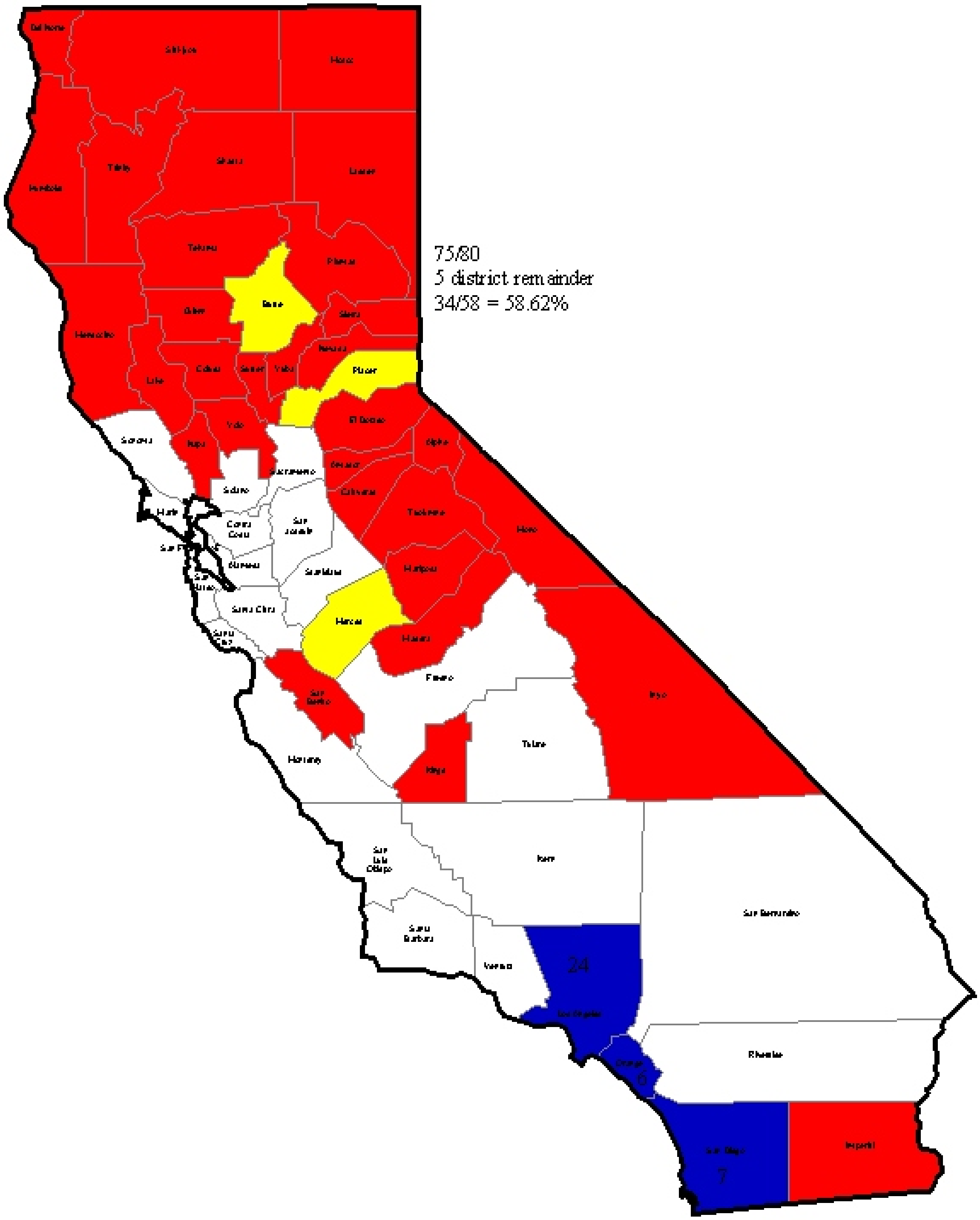
Proposition 49.0 (District Allocation Failure I) Too coarse of a partition ⊑ regional district with a large number of counties and a multi-county division district solution.

Proposition 50.0 (District Allocation Failure II) Too fine of a partition ⊑ large number of county subdivision districts in the ($J = \emptyset$) absence of organized local jurisdiction, by unorganized county territory, unincorporated UMSAs (unincorporated municipal service areas of county territory), CDPs (census designated places) and other urban (zonal) areas, places with few incorporated cities or only minor incorporated civil jurisdictions.

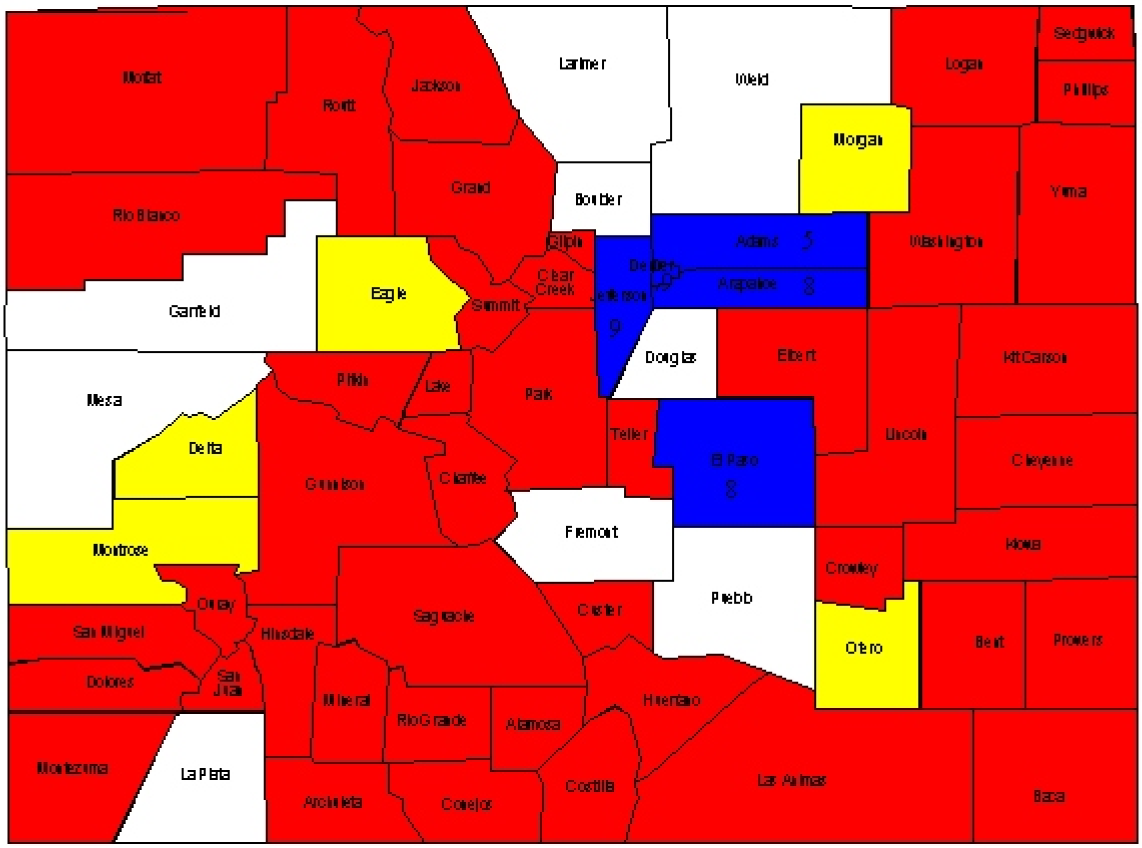
Appendix I

61/60
-1 district remainder
Maricopa = 34 + 1 = Maricopa + La Paz
Pima = 10 + 1 = Pima + Santa Cruz
Graham + Greenlee = 1 + 61 = 62
4/15 = 26.67%

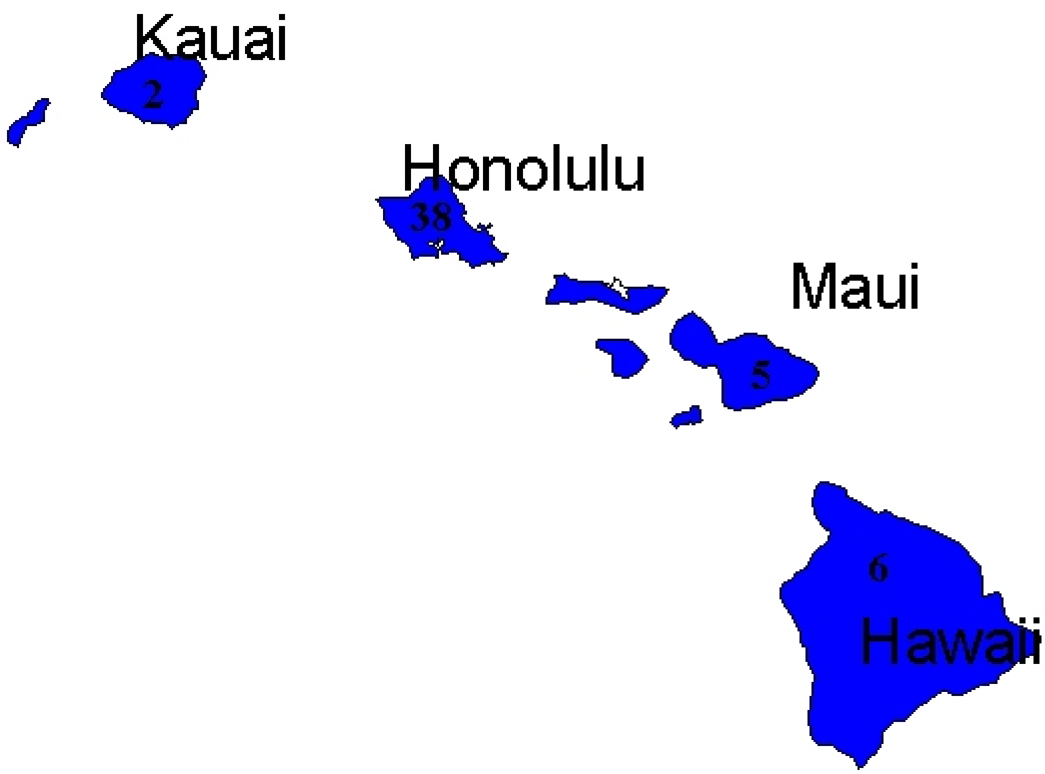



75/80
5 district remainder
34/58 = 58.62%
24
6
7

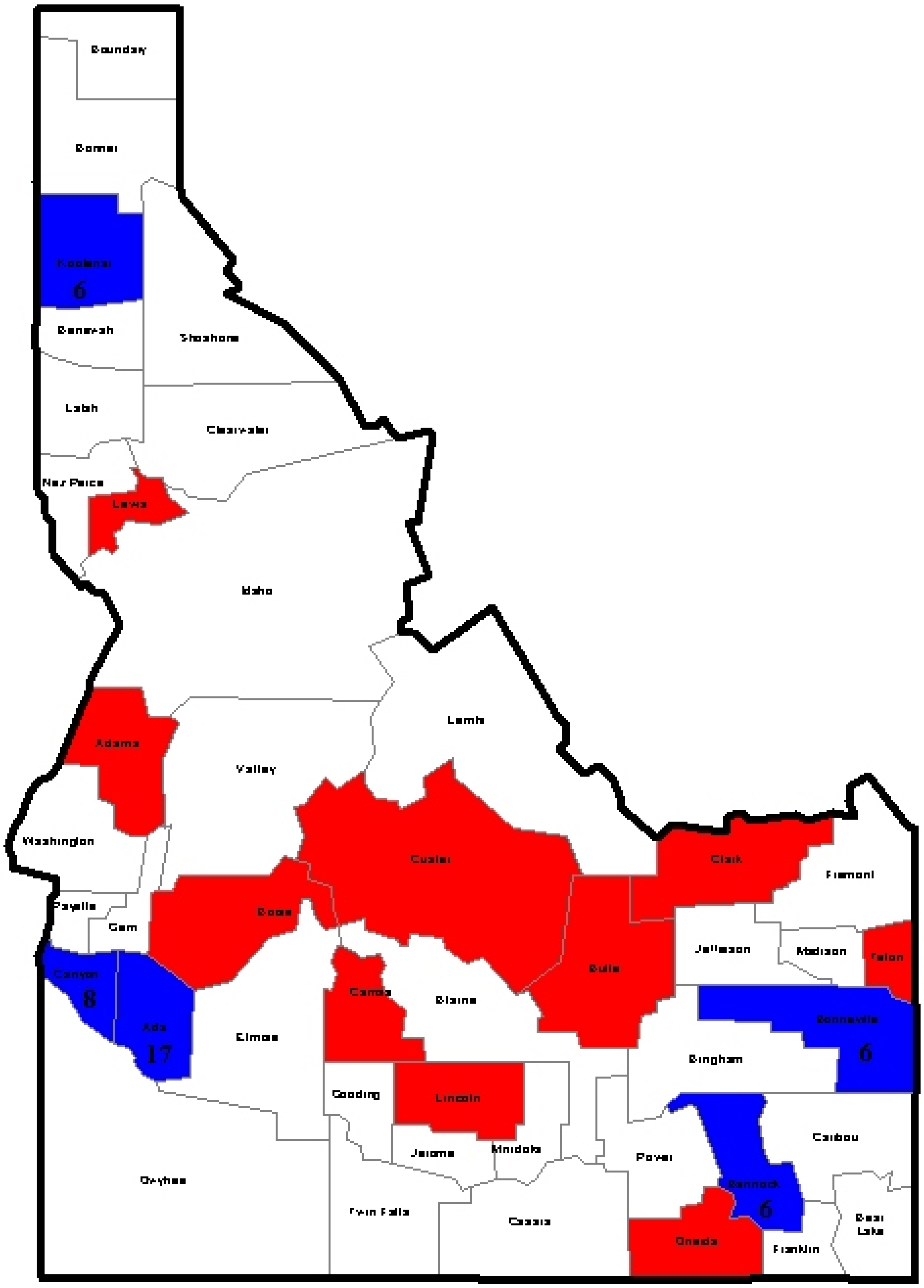
58/65
7 district remainder
49/63 = 77.78%



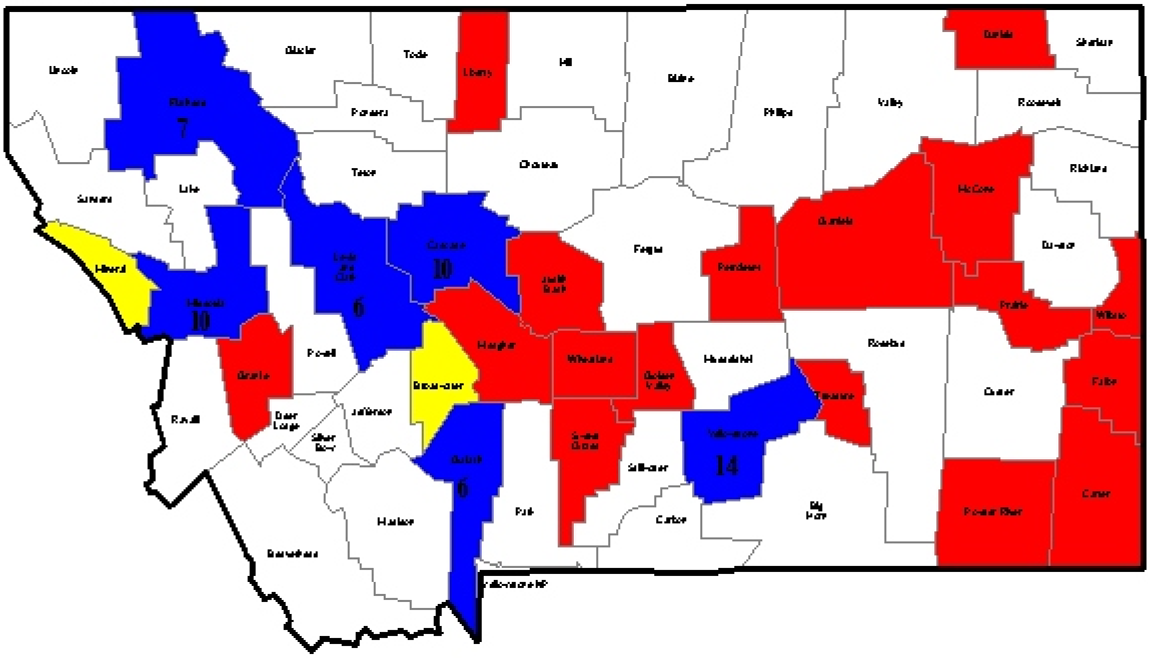
51/51
0 district remainder
0 / 4 = 0.0%



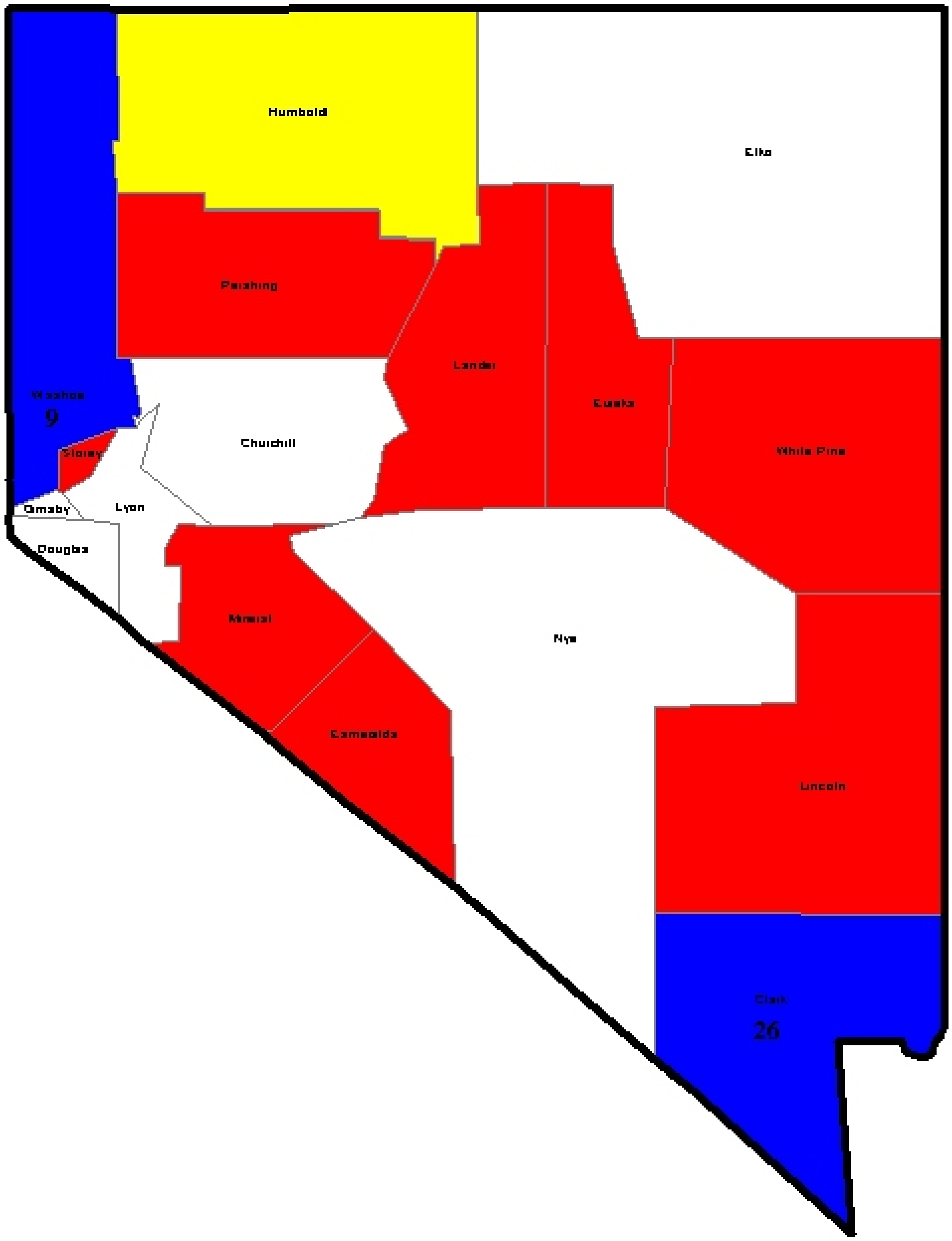
86/84
-2 district remainder
10/44 = 22.73%



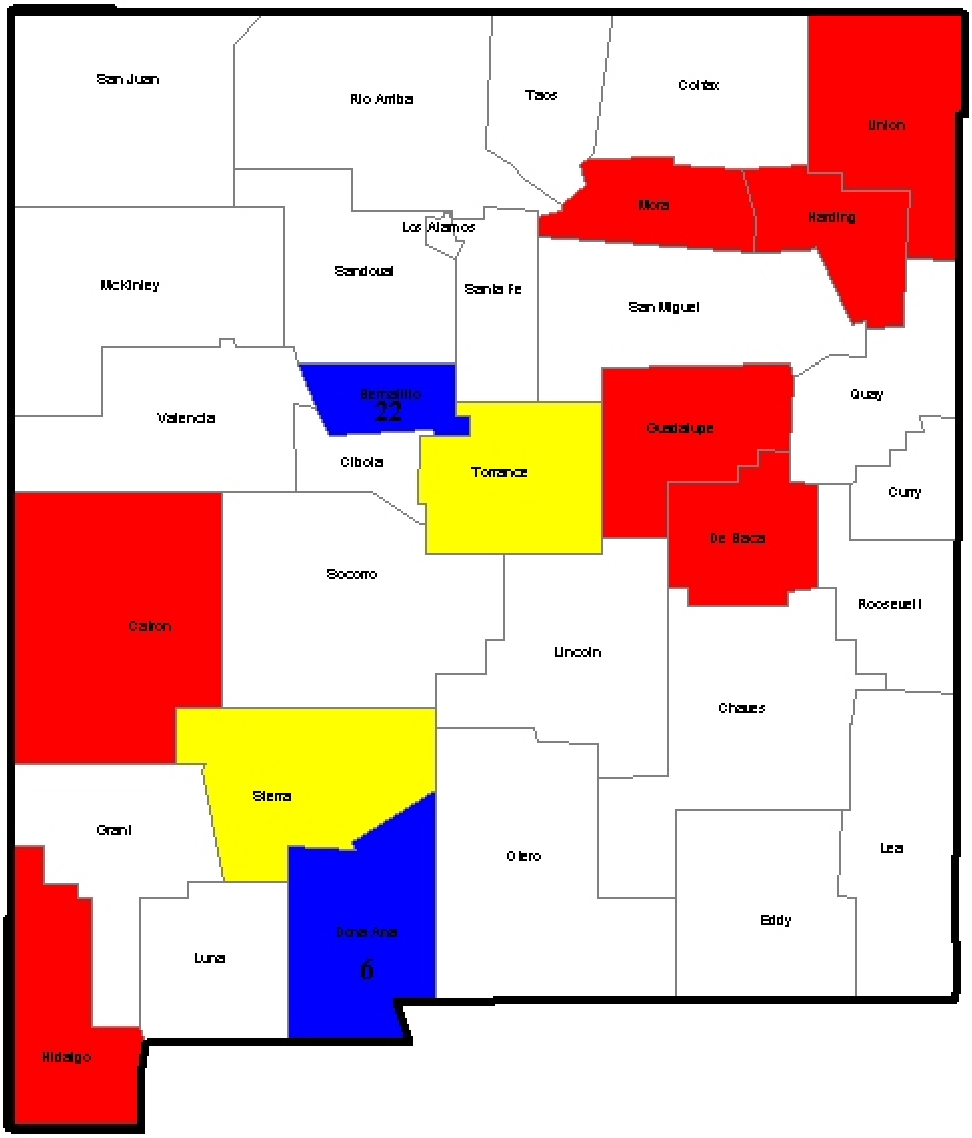
96/100
4 district remainder
19/56 = 33.93%



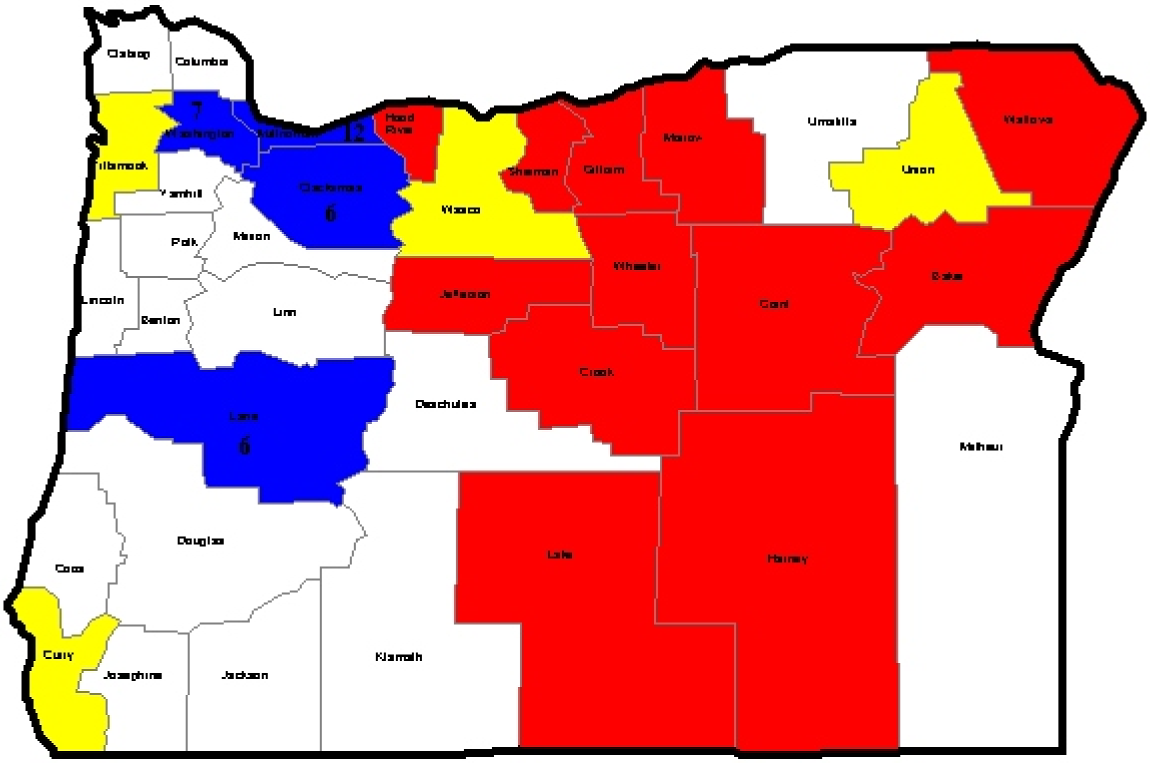
41/42
1 district remainder
9/17 = 52.94%



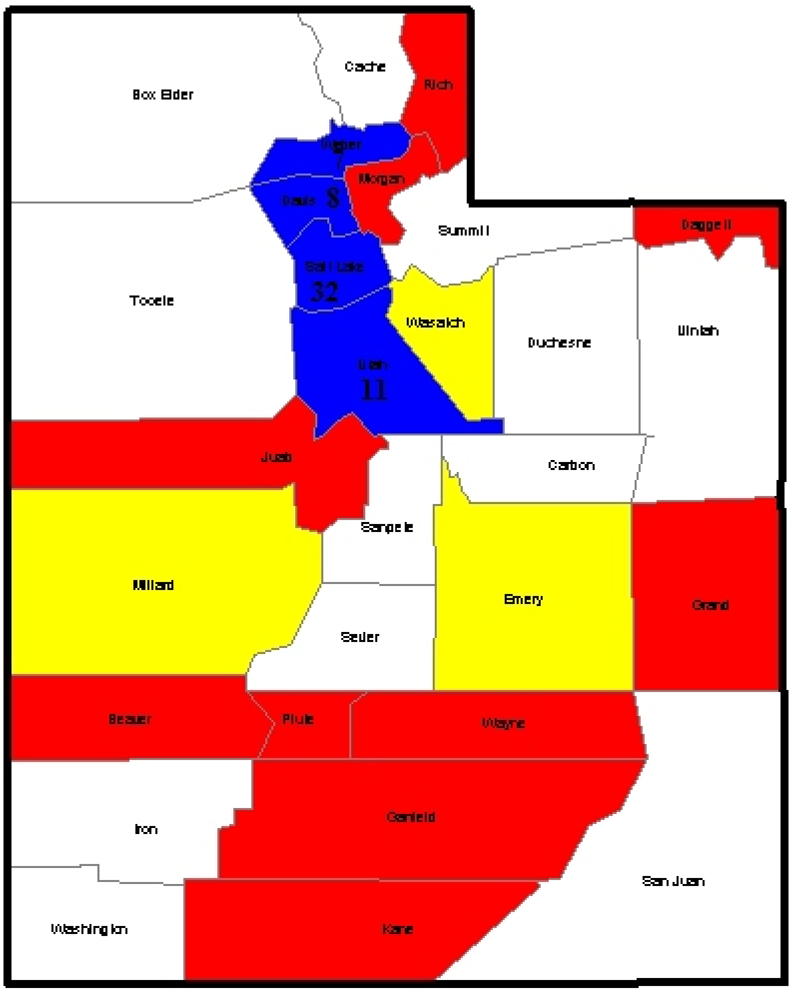
70/70
0 district remainders
9/33 = 27.27%

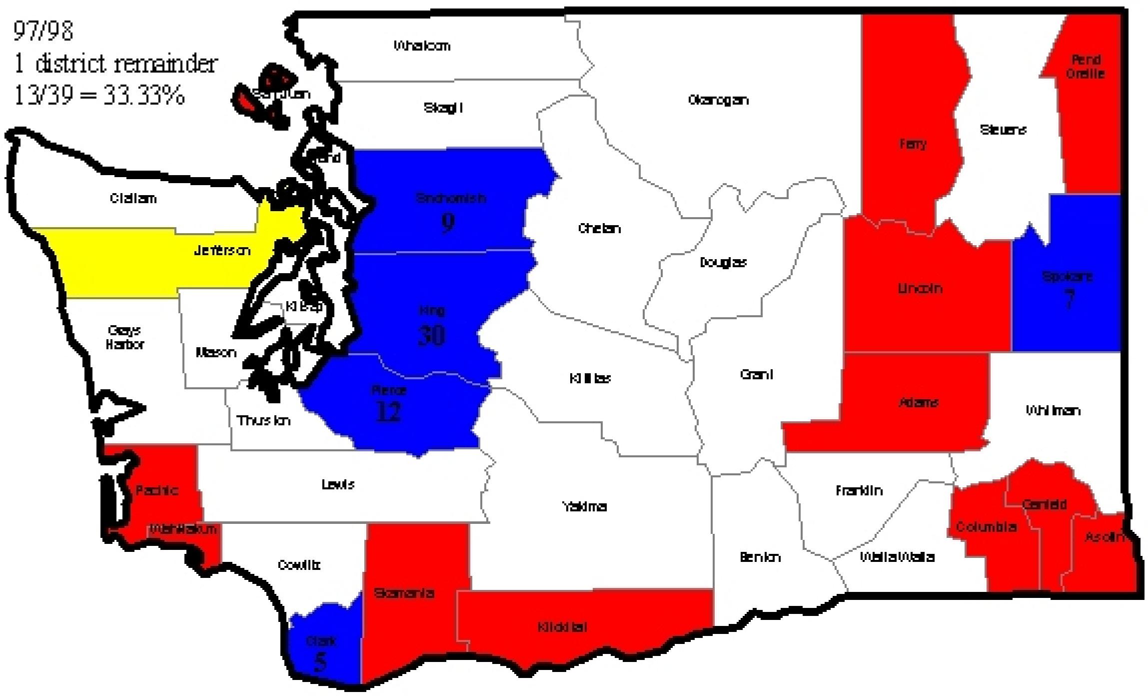


56/60
4 District remainder
16/36 = 44.44%

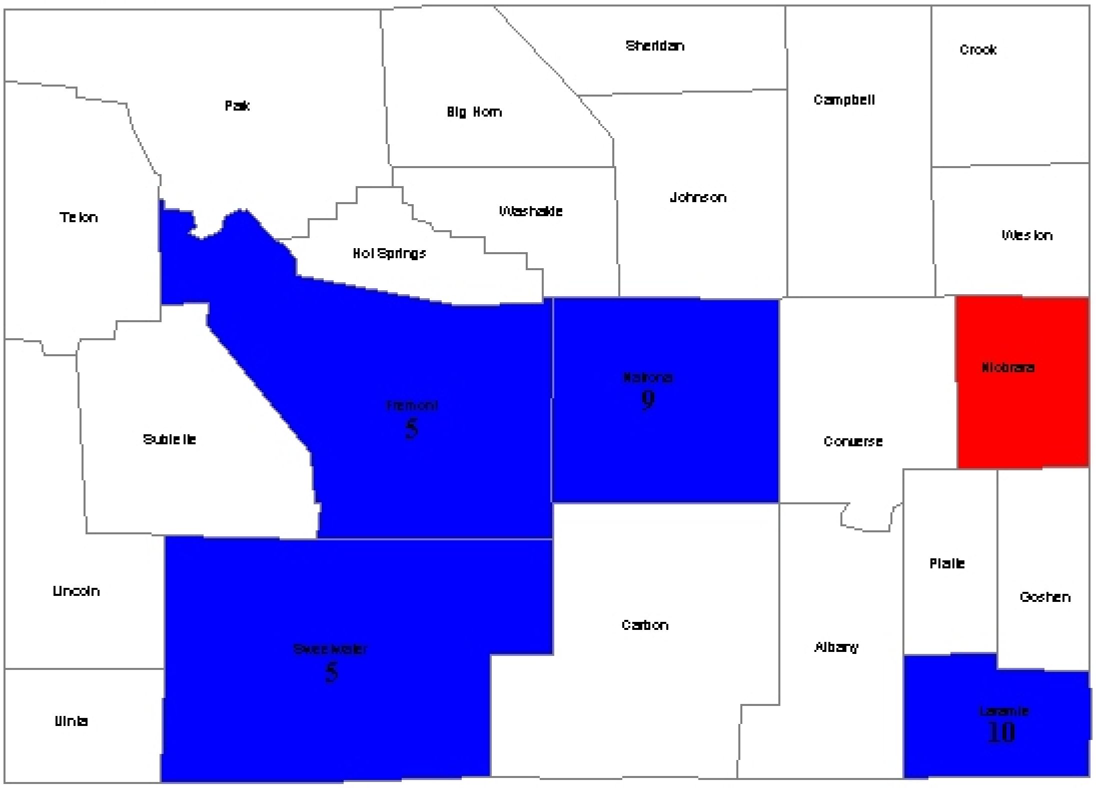


74/75
1 district remainder
13/29 = 48.83%




97/98
1 district remainder
13/39 = 33.33%
Whatcom
Skagit
Okanogan
Ferry
Stevens
Pend Oreille
Clallam
Snohomish
9
Chelan
Jefferson
Douglas
Lincoln
Spokane
7
King
30
Grays Harbor
Mason
Pierce
12
Kittitas
Grant
Adams
Whitman
Thurston
Pacific
Lewis
Yakima
Franklin
Garfield
Columbia
Asotin
Wahkiakum
Cowlitz
Benton
Walla Walla
Skamania
Klickitat
Clark
5

64/64
0 district remainder
1/23 = 4.35%



Appendix II

TABLE 1.0 Largest District Senate Apportionment by Population Ratios from 1900 Census

State	Local Jurisdiction	Senate Population Ratio	State	Local Jurisdiction	Senate Population Ratio
RI	Providence	28.377	NH	Grafton	2.382
IL	Cook	19.449	GA	Fulton	2.330
NY	New York	14.105	VT	Windham	2.327
MD	Baltimore City	11.138	NH	Strafford	2.294
PA	Philadelphia	10.264	CO	Pueblo	2.234
DE	New Castle	10.095	CT	New London	2.186
CO	Arapahoe	9.923	FL	Hillsborough	2.180
CA	San Francisco	9.233	MA	Norfolk	2.161
MA	Suffolk	8.718	MO	Jackson	2.136
MN	Hennepin	8.214	VT	Caledonia	2.128
LA	Orleans	8.104	UT	Utah	2.111
MA	Middlesex	8.066	SD	Lawrence	2.089
NY	Kings	8.024	RI	Washington	2.085
OR	Multnomah	7.484	CO	El Paso	2.049
WA	King	7.222	PA	Luzerne	2.040
CT	New Haven	7.111	HI	Kauai	2.020
NH	Hillsborough	6.568	ME	Washington	2.019
MO	Saint Louis City	6.296	NV	Elko	2.015
PA	Allegheny	6.149	OR	Marion	2.010
MN	Ramsey	6.135	NE	Lancaster	2.007
HI	Honolulu	5.698	TN	Davidson	2.006
WI	Milwaukee	5.264	SC	Spartanburg	2.005
CT	Hartford	5.164	KS	Wyandotte	1.992
MA	Essex	5.091	MD	Baltimore	1.986

UT	Salt Lake	5.055	ND	Walsh	1.984
MA	Worcester	4.947	WY	Carbon	1.969
CT	Fairfield	4.867	FL	Alachua	1.952
MT	Silver Bow	4.750	VT	Orleans	1.923
MI	Wayne	4.610	VT	Addison	1.913
CA	Los Angeles	4.587	MT	Lewis & Clark	1.912
HI	Hawaii	4.563	VT	Bennington	1.895
ME	Cumberland	4.495	CO	Teller	1.881
NE	Douglas	4.351	IA	Polk	1.851
WY	Laramie	4.144	VA	Richmond	1.835
KY	Jefferson	4.116	NH	Cheshire	1.826
NJ	Essex	4.003	SC	Orangeburg	1.825
IN	Marion	3.919	ID	Nez Perce	1.785
DE	Sussex	3.890	SD	Brown	1.784
VT	Rutland	3.859	ND	Pembina	1.748
WA	Spokane	3.776	ID	Latah	1.746
WA	Pierce	3.643	WY	Sweetwater	1.736
MA	Bristol	3.594	MT	Deer Lodge	1.734
CA	Alameda	3.507	NJ	Passaic	1.730
VT	Chittenden	3.457	NH	Coos	1.718
ME	Penobscot	3.404	MI	Kent	1.715
OH	Cuyahoga	3.274	FL	Escambia	1.714
NV	Washoe	3.239	SC	Anderson	1.705
VT	Washington	3.196	ND	Richland	1.701
NH	Merrimack	3.057	VT	Orange	1.686
OH	Hamilton	3.053	CT	Litchfield	1.682
DE	Kent	3.015	ID	Fremont	1.664
MN	Saint Louis	2.983	WA	Whitman	1.664

NY	Erie	2.983	ME	Hancock	1.662
NH	Rockingham	2.981	UT	Weber	1.642
ME	York	2.896	SC	Greenville	1.636
RI	Newport	2.814	MN	Otter Tail	1.632
VT	Windsor	2.813	MA	Plymouth	1.625
ND	Cass	2.800	CA	Santa Clara	1.622
SD	Minnehaha	2.793	MN	Stearns	1.599
ME	Aroostook	2.712	WA	Whatcom	1.583
SC	Charleston	2.692	NV	Humboldt	1.581
WY	Albany	2.687	WA	Snohomish	1.572
ME	Kennebec	2.639	SC	Sumter	1.567
VT	Franklin	2.636	ID	Shoshone	1.551
HI	Maui	2.605	AR	Pulaski	1.541
RI	Kent	2.588	SD	Turner	1.538
MT	Cascade	2.571	PA	Lackawanna	1.538
AL	Jefferson	2.534	IN	Allen	1.535
WY	Uinta	2.510	MS	Hinds	1.525
TN	Shelby	2.508	ID	Bannock	1.519
MA	Hampden	2.504	ME	Somerset	1.511
ME	Androscoggin	2.421	ID	Ada	1.501
FL	Duval	2.406	NY	Monroe	1.499
ND	Grand Forks	2.392	LA	Saint Landry	1.493

TABLE 2.0 Reapportionment in changes of the size of The State Legislatures

STATE	Mean	N	Std. Deviation	Median	Std. Error of Mean	House+ Senate = Sum	Skewness	Kurtosis
AK	12.00	3	4.00	12.00	2.31	36	.000	.
AL	3.50	2	2.12	3.50	1.50	7	.	.
AR	3.00	1	.	.	.	3	.	.
AZ	5.22	9	14.02	9.00	4.67	47	-.525	.208
CT	-9.20	10	39.48	4.50	12.49	-92	-2.747	7.871
DE	2.50	4	1.00	2.00	.50	10	2.000	4.000
FL	8.57	7	10.55	9.00	3.99	60	-.097	.000
GA	1.70	10	10.13	1.50	3.20	17	.346	1.253
HI	15.50	2	7.78	15.50	5.50	31	.	.
IA	.00	2	11.31	.00	8.00	0	.	.
ID	6.22	9	9.31	7.00	3.10	56	-.230	-.225
IL	-6.75	4	36.17	4.00	18.08	-27	-1.557	2.834
LA	1.17	6	10.74	3.00	4.38	7	-.658	2.037
MA	-80.00	1	.	.	.	-80	.	.
MD	6.64	11	6.93	4.00	2.09	73	.779	-.950
ME	1.00	4	1.41	1.50	.71	4	-1.414	1.500
MI	5.33	3	4.16	4.00	2.40	16	1.293	.
MN	3.17	6	3.87	2.50	1.58	19	1.179	1.646
MO	4.60	5	2.41	4.00	1.08	23	.601	-.945
MS	-1.00	4	11.46	3.50	5.73	-4	-1.866	3.603
MT	4.67	12	13.45	2.50	3.88	56	1.296	1.238
ND	8.25	8	15.98	4.50	5.65	66	1.132	2.512
NE	8.00	2	2.83	8.00	2.00	16	.	.
NH	.50	6	13.55	3.50	5.53	3	-1.390	2.028
NJ	19.50	2	.71	19.50	.50	39	.	.
NM	10.67	6	6.98	9.00	2.85	64	1.772	3.743
NV	1.55	11	7.57	3.00	2.28	17	.371	2.145
NY	1.83	6	2.04	1.50	.83	11	.333	.516
OH	-.64	14	12.29	3.00	3.28	-9	-2.816	9.177
OK	4.25	8	12.13	4.00	4.29	34	-1.518	3.365
PA	-.25	4	4.57	1.50	2.29	-1	-1.811	3.380
RI	6.83	6	10.53	2.50	4.30	41	2.288	5.336
SC	2.50	2	.71	2.50	.50	5	.	.
SD	-6.40	5	16.73	-5.00	7.48	-32	.201	.364
TX	4.40	5	4.51	5.00	2.01	22	-.600	-.942
UT	3.73	11	2.10	3.00	.63	41	.755	.039
VT	-24.00	4	48.02	-.50	24.01	-96	-1.995	3.982
WA	5.33	6	6.41	3.00	2.62	32	1.471	2.204
WI	-1.00	1	.	.	.	-1	.	.
WV	6.17	6	4.92	5.00	2.01	37	1.365	1.810
WY	3.70	10	6.72	3.00	2.12	37	.823	1.393
Total	2.47	238	15.46	3.00	1.00	588	-3.935	25.060

TABLE 3.0 Single County Senate Apportionment, Population Ratios from 1900 Census

State	Local Jurisdiction	Senate Population Ratio	State	Local Jurisdiction	Senate Population Ratio
WV	Kanawha	1.483	AL	Mobile	1.132
FL	Marion	1.477	OR	Baker	1.131
SD	Yankton	1.476	KY	Kenton	1.125
SD	Brookings	1.466	MN	Goodhue	1.120
KS	Shawnee	1.461	LA	Rapides	1.117
NC	Mecklenburg	1.459	KS	Leavenworth	1.114
ME	Oxford	1.439	NJ	Union	1.108
SD	Day	1.430	WI	Dane	1.107
MS	Washington	1.426	SD	Spink	1.107
OR	Clackamas	1.426	MS	Lauderdale	1.107
IN	Vanderburgh	1.426	CT	Middlesex	1.103
SD	Roberts	1.426	PA	Montgomery	1.103
OR	Lane	1.422	WY	Fremont	1.100
SD	Lincoln	1.419	VA	Norfolk	1.095
CO	Las Animas	1.416	MT	Chouteau	1.094
FL	Jackson	1.415	FL	Monroe	1.090
GA	Chatham	1.414	CO	Weld	1.090
IN	Madison	1.400	SD	Clay	1.087
CO	Boulder	1.397	IN	Grant	1.087
SC	Richland	1.395	SC	Barnwell	1.086
MT	Missoula	1.393	SC	Beaufort	1.086
SD	Hutchison	1.389	ME	Waldo	1.080
PA	Schuylkill	1.372	SC	Marion	1.076
MA	Berkshire	1.364	MI	Saginaw	1.074
ME	Knox	1.357	VT	Lamoille	1.073
ID	Bingham	1.356	MS	Holmes	1.068
OR	Linn	1.350	GA	Richmond	1.067

MO	Buchanan	1.333	SD	Lake	1.067
ID	Kootenai	1.326	NC	Robeson	1.066
OR	Umatilla	1.309	NJ	Mercer	1.063
WV	Ohio	1.302	SD	Grant	1.063
SD	Union	1.302	UT	Sanpete	1.061
NV	Storey	1.301	OR	Douglas	1.057
AL	Montgomery	1.300	KS	Crawford	1.056
ND	Barnes	1.287	NY	Queens	1.052
MN	Winona	1.284	WY	Sheridan	1.052
ND	Trail	1.282	NH	Sullivan	1.050
SC	York	1.275	OR	Washington	1.050
MS	Yazoo	1.275	MN	Wright	1.049
MN	Polk	1.274	CA	Sonoma	1.036
PA	Westmoreland	1.271	FL	Columbia	1.035
NY	Westchester	1.267	NC	Guilford	1.032
PA	Berks	1.266	MS	Bolivar	1.028
IA	Dubuque	1.264	NV	Carson City	1.025
PA	Lancaster	1.263	SD	Codington	1.024
LA	Caddo	1.256	SC	Colleton	1.023
IA	Linn	1.241	SC	Abbeville	1.022
CT	Windham	1.238	CA	Fresno	1.020
CA	Sacramento	1.237	MN	Fillmore	1.016
IN	Vigo	1.233	SD	McCook	1.014
ND	Cavalier	1.230	CO	Fremont	1.014
OH	Franklin	1.226	VA	Pittsylvania	1.012
WA	Walla Walla	1.226	TN	Hamilton	1.008
IA	Woodbury	1.223	VA	Norfolk	1.006
TN	Knox	1.213	GA	Bibb	1.002
SD	Bon Homme	1.211	MN	Washington	1.000
FL	Leon	1.204	AR	Jefferson	1.000

NJ	Camden	1.200	MS	Copiah	.998
KS	Sedgwick	1.198	WA	Lewis	.995
SC	Aiken	1.194	OR	Jackson	.994
MS	Warren	1.187	WA	Grays Harbor	.992
ID	Idaho	1.184	SD	Charles Mix	.992
UT	Cache	1.180	SC	Darlington	.991
MD	Allegany	1.175	MD	Washington	.988
CO	Lake	1.171	AL	Dallas	.986
IN	Saint Joseph	1.170	IN	Delaware	.986
NC	Buncombe	1.169	NH	Carroll	.985
OR	Union	1.166	IA	Clinton	.982
NV	Lincoln	1.164	FL	Jefferson	.981
KS	Cherokee	1.161	OR	Yamhill	.974
NY	Onondaga	1.161	ID	Canyon	.973
MN	Blue Earth	1.161	SD	Moody	.972
ID	Oneida	1.160	OH	Montgomery	.970
IA	Scott	1.155	SC	Williamsburg	.969
SD	Kingsbury	1.152	LA	Saint Mary	.964
OH	Lucas	1.145	KY	Campbell	.960
SC	Laurens	1.144	OR	Wasco	.985
NY	Albany	1.139	CA	San Joaquin	.955
NH	Belknap	1.139	MT	Gallatin	.953
MD	Frederick	1.136	CA	San Diego	.945
RI	Bristol	1.135	SD	Beadle	.943